SPINORS, NONLINEAR CONNECTIONS, AND NEARLY AUTOPARALLEL MAPS OF GENERALIZED FINSLER SPACES

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Abstract. We study the geometric setting of the field theory with locally anisotropic interactions. The concept of locally anisotropic space is introduced as a general one for various type of extensions of Lagrange and Finsler geometry and higher dimension (Kaluza–Klein type) spaces. The problem of definition of spinors on generalized Finsler spaces is solved in the framework of the geometry of Clifford bundles provided with compatible nonlinear and distinguished connections and metric structures. We construct the spinor differential geometry of locally anisotropic spaces and discuss some related issues connected with the geometric aspects of locally anisotropc interactions for gravitational, gauge, spinor, Dirac spinor and Proca fields. Motion equations in generalized Finsler spaces, of the mentioned type of fields, are defined in a geometric manner by using bundles of linear and affine frames locally adapted to the nonlinear connection structure. The nearly autoparallel maps are introduced as maps with deformation of connections extending the class of geodesic and conformal transforms. Using this approach we propose two variants of solution of the problem of definition of conservation laws on locally anisotropic spaces.

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PREFACE

The generalizations of Finsler spaces [Finsler 1918] (see fundamental contributions and references in [Cartan 1934] and [Rund 1959]) with various applications in physics, chemistry, biology, ecology etc have been a subject for extensive study in the last two decades [Miron and Anastasiei 1987, 1994], [Asanov 1985], [Matsumoto 1986], [Bejancu 1990], [Miron and Kavaguchi 1996] and [Antonelli and Miron (eds) 1996]. There were developed a number of theories of Finsler gravity and gauge fields or of stochastic processes in generalized Finsler spaces. Recently, a deal of attention is attracted by the problem of formulating of the geometric background of classical and quantum field interactions with local anisotropy. We note some of our contributions based on modeling of locally anisotropic physical theories on vector bundles provided with nonlinear connection structure [Vacaru 1996], [Vacaru and Goncharenko 1995] and [Vacaru and Ostaf 1996a] which make up a starting point of this work. Our approach stands out because it allows a rigorous definition of spinors in generalized Finsler spaces and a corresponding geometric treatment of fundamental fields interactions on such spaces.

Spinor variables on Finsler spaces have been considered in a heuristic manner, for instance, by [Asanov and Ponomarenko 1988] and [Ono and Takano 1993]. A substantial difficulty that arises in every attempt to construct physical models on Finsler spaces is the absence of groups of automorphisms even locally on spaces with generic local anisotropy and the existence of a new fundamental geometric object the nonlinear connection. In consequence it is a chalenging task to introduce spinors as Clifford structures or to define conservation laws by using usual variational calculus. Our key idea was to concentrate efforts not on definition of various types of spinor spaces for every particular class of Finsler, and theirs generalizations, spaces but to apply the fact that spaces with local anisotropy can be in general modeled on tangent [Yano and Ishihara 1973] or vector [Miron and Anastasiei 1987, 1994] bundles provided with nonlinear and distinguished connections and metric structures. If the mentioned connections and metric structure are compatible, the Clifford and spinor locally anisotropic bundles can be introduced similarly as for curved spaces or vector bundles [Geroch 1958], [Karoubi 1978] and [Turtoi 1989] with that distinction that the geometric constructions must be adapted to the nonlinear connection (see a study of Clifford and spinor structures on generalized Lagrange and Finsler spaces in [Vacaru 1996] and an geometric approach to locally anisotropic gauge fields and gauge like gravity in [Vacaru and Goncharenko 1995]).

Regarding the definition of conservation laws on locally anisotropic spaces, we

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propose to introduce into consideration a new class of maps with deformation of connections of both type of locally isotropic or anisotropic curved spaces. We are inspired by the idea [Petrov 1970] (see also H. Poincare works [Poincare 1905, 1954] on conventionality of concepts of geometrical space–time and physical theories) that geometric constuctions and physical models can be locally equivalently modeled on arbitrary curved, or (if some well defined conditions are satisfied) flat space by using corresponding generalizations of conformal transforms. Partially the geometric aspect of the Petrov's program on modeling of field interactions was realized in the monograph [Sinyukov 1979] where the theory of nearly geodesic maps of Riemannian and affine connection spaces is formulated. Nearly geodesic maps generalizes the class of conformal, geodesic and concircular transforms [Schouten and Struik 1938], [Yano 1940], [Vrânceanu 1977] and [Mocanu 1955]. We extended (see [Vacaru 1987], [Vacaru 1992], [Vacaru 1994] and [Vacaru and Ostaf 1996b]) the Sinyukov's theory for spaces with torsion and nonmetricity, the so–called Einstein–Cartan–Weyl spaces, by considering nearly autoparallel maps, developed some approaches to formulation of conservation laws for physical interactions and definition of twistors [Penrose and Rindler 1986] on such spaces and proposed a nearly autoparallel map classification of Lagrange and Finsler space in paper [Vacaru and Ostaf 1996a].

After presenting this informal discussion of some basic ideas and results to be used in our further considerations, we now turn to a more detailed description of the content of this work.

The presentation in the Chapter I is organized as follows: The geometry of nonlinear connections in vector bundles is revewed in section I.1, were the explicit formulas for torsions and curvatures on locally anisotropic spaces are given and motion equations for fundamental field interactions with local anisotropy are introduced in a geometric manner. Section I.2 is devoted to the distinguished Clifford algebras. Clifford bundles and distinguished by nonlinear connections spinor structures are defined in section I.3. Almost complex spinor structures for generalized Lagrange spaces are analyzed in section I.4. In section I.5 the spinor techniques is developed for distinguished vector spaces. The differential geometry of locally anisotropic spinors is considered in section I.6 where distinguished spinor formulas for connections torsions and curvatures are presented. The spinor form of field equations on locally anisotropic spaces is analyzed in section I.7.

The Capter II is devoted to the geometry of gauge fields and gauge gravity in locally anisotropic spaces. In section II.1 we give a geometrical interpretation of gauge (Yang-Mills) fields on general locally anisotropic spaces. Section II.2 contains a geometrical definition of anisotropic Yang-Mills equations; the variational proof of gauge field equations is considered in connection with the "pure" geometrical method of introducing field equations. In section II.3 the locally anisotropic gravity is reformulated as a gauge theory for nonsemisimple groups. A model of nonlinear de Sitter gauge gravity with local anisotropy is formulated in section II.4.

The problem of formulation of conservation laws on locally anisotropic spaces is investigated in the framework of the geometry of local 1–1 maps of vector bundles provided with nonlinear connection structures and by developing the formalism of tensor–integral for locally anisotropic multispaces in the Chapter III. Section III.1 is devoted to the formulation of the theory of nearly autoparallel maps of locally

anisotropic spaces. The classification of na–maps and formulation of their invariant conditions are given in section III.2. In section III.3 we define the nearly autoparallel tensor–integral on locally anisotropic multispaces. The question of formulation of conservation laws on spaces with local anisotropy is studied in section III.4. We present a definition of conservation laws for locally anisotropic gravitational fields on nearly autoparallel images of locally anisotropic spaces in section III.5.

CHAPTER I

LOCALLY ANISOTROPIC SPINOR SPACES

The purpose of this Chapter is to present an introduction into the geometry of spinors in generalized Finsler spaces and to propose a geometric framework for the theory field interactions on locally anisotropic (la) spaces starting from papers [Vacaru 1996] and [Vacaru and Goncharenko 1995] (in brief we shall write la–spaces, la–geometry, la–spinors and so on). The geometric constructions will be adapted to the nonlinear connection (N–connection) structure. We consider the reader to be familiar with basic results on differential geometry of bundle spaces [Bishop and Crittenden 1964] and [Kobayashi and Nomizu 1963, 1969] and note that as a rule all geometric constructions in this work will be local in nature.

For our considerations on the la–spinor theory it will be convenient to extend the [Penrose and Rindler 1984, 1986] abstract index approach (see also the [Luehr and Rosenbaum 1974] index free methods) proposed for spinors on locally isotropic spaces. We note that for applications in mathematical physics we usually we have dimensions $d > 4$ for spaces into consideration. In this case the 2–spinor calculus does not play a preferential role.

I.1 Connections and Metrics in Locally Anisotropic Spaces

As a preliminary to a discussion of la–spinor formalism we review some results and methods of the differential geometry of tangent and vector bundles provided with nonlinear and distinguished connections and metric structures (see regorous results and details in [Miron and Anastasiei 1987, 1994] and [Yano and Ishihara 1973]). This section serves the twofold purpose of establishing of abstract index denotations and starting the geometric backgrounds which are used in the next sections of the Chapter. Combersome proofs and calculations will not be presented.

I.1.1 Nonlinear connections and distinguishing of geometric objects.

Let $\mathcal{E} = (E, p, M, Gr, F)$ be a locally trivial vector bundle, v–bundle, where $F = \mathbb{R}^m$ (a real vector space of dimension $m, \dim F = m$, $\mathbb{R}$ denotes the real number field) is the typical fibre, the structural group is chosen to be the group of automorphisms of $\mathbb{R}^m$, i.e. $Gr = GL(m, \mathbb{R})$, and $p : E \to M$ is a differentiable surjection of a differentiable manifold E (total space, $\dim E = n+m$) to a differentiable manifold M (base space, $\dim M = n$) . Local coordinates on $\mathcal{E}$ are denoted as $u^{\boldsymbol{\alpha}} = \left(x^{\mathbf{i}}, y^{\mathbf{a}}\ \right)$, or in brief $\mathbf{u} = (x, y)$, where boldfaced indices will be considered as coordinate ones

for which the Einstein summation rule holds (Latin indices $\mathbf{i}$,$\mathbf{j}$,$\mathbf{k}$,... $= 1, 2, ..., n$ will parametrize coordinates of geometrical objects with respect to a base space M, Latin indices $\mathbf{a}$,$\mathbf{b}$,$\mathbf{c}$,... $= 1, 2, ..., m$ will parametrize fibre coordinates of geometrical objects and Greek indices $\boldsymbol{\alpha}$,$\boldsymbol{\beta}$, $\boldsymbol{\gamma}$,... are considered as cumulative ones for coordinates of objects defined on the total space of a v-bundle). We shall correspondingly use abstract indices $\alpha = (i, a)$, $\beta = (j, b), \gamma = (k, c), ...$ in the Penrose manner [Penrose and Rindler 1984, 1986] in order to mark geometical objects and theirs (base, fibre)-components or, if it will be convenient, we shall consider boldfaced letters (in the main for pointing to the operator character of tensors and spinors into consideration) of type $\mathbf{A} \equiv A = \left(A^{(h)}, A^{(v)}\right), \mathbf{b} = \left(b^{(h)}, b^{(v)}\right), ..., \mathbf{R}, \boldsymbol{\omega}, \boldsymbol{\Gamma}, ...$ for geometrical objects on $\mathcal{E}$ and theirs splitting into horizontal (h), or base, and vertical (v), or fibre, components. For simplicity, we shall prefer writing out of abstract indices instead of boldface ones if this will not give rise to ambiguities.

Coordinate transforms $u^{\boldsymbol{\alpha}'} = u^{\boldsymbol{\alpha}'}\ (u^{\boldsymbol{\alpha}})$ on $\mathcal{E}$ are written as

$$\left(x^{\mathbf{i}}, y^{\mathbf{a}}\right) \rightarrow \left(x^{\mathbf{i}'}\ , y^{\mathbf{a}'}\right),$$

where

$$x^{\mathbf{i}'}\ = x^{\mathbf{i}'}\ (x^{\mathbf{i}}), y^{\mathbf{a}'}\ = M^{\mathbf{a}'}_{\mathbf{a}}(x^{\mathbf{i}}\ )y^{\mathbf{a}}$$

and matrices $M^{\mathbf{a}'}_{\mathbf{a}}(x^{\mathbf{i}}\ ) \in GL\,(m, \mathbb{R})$ are functions of necessary smoothness class.

A local coordinate parametrization of $\mathcal{E}$ naturally defines a coordinate basis

$$\frac{\partial}{\partial u^\alpha} = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a}\right), \tag{1.1}$$

in brief we shall write $\partial_\alpha = (\partial_i, \partial_a)$, and the reciprocal to (1.1) coordinate basis

$$du^\alpha = (dx^i, dy^a), \tag{1.2}$$

or, in brief, $d^\alpha = (d^i, d^a)$, which is uniquely defined from the equations

$$d^\alpha \circ \partial_\beta = \delta^\alpha_\beta,$$

where δ^α_β is the Kronecher symbol and by "$\circ$" we denote the inner (scalar) product in the tangent bundle $\mathcal{T}E$.

The concept of **nonlinear connection,** is fundamental in the geometry of la–spaces. It came from Finsler geometry, [Cartan 1934] and [Kawaguchi 1952, 1956], and was globally defined by [Barthel 1963] (see a detailed study and basic references in [Miron and Anastasiei 1987, 1994]).

In a v–bundle $\mathcal{E}$ we can consider the distribution $\{N : E_u \rightarrow H_u E, T_u E = H_u E \oplus V_u E\}$ on E being a global decomposition, as a Whitney sum, into horizontal,$\mathcal{H}E$, and vertical, $\mathcal{V}E$, subbundles of the tangent bundle $\mathcal{T}E$:

$$\mathcal{T}E = \mathcal{H}E \oplus \mathcal{V}E. \tag{1.3}$$

Definition 1.1. *A nonlinear connection in the vector bundle* (E, p, M) *is a splitting on the left of the exact sequence*

$$0 \longrightarrow \quad VE \quad \stackrel{i}{\longrightarrow} \quad TE \quad \longrightarrow \quad TE/VE \quad \longrightarrow 0$$

that is a morphism of vector bundles $C : TE \to VE$ *such that* $C \circ i$ *is the identity on* VE.

Locally a N–connection in $\mathcal{E}$ is given by it components $N_{\mathbf{i}}^{\mathbf{a}}(\mathbf{u}) = N_{\mathbf{i}}^{\mathbf{a}}(\mathbf{x}, \mathbf{y})$ (in brief we shall write $N_i^a(u) = N_i^a(x, y)$) with respect to bases (1.1) and (1.2)):

$$\mathbf{N} = N_i^a(u) d^i \otimes \partial_a.$$

We note that a linear connection in a v-bundle $\mathcal{E}$ can be considered as a particular case of a N–connection when $N_i^a(x, y) = K_{bi}^a(x)\, y^b$, where functions $K_{ai}^b(x)$ on the base M are called the Christoffel coefficients.

To coordinate locally geometric constructions with the global splitting of $\mathcal{E}$ defined by a N-connection structure, we have to introduce a locally adapted basis (la–basis, la–frame):

$$\frac{\delta}{\delta u^\alpha} = \left(\frac{\delta}{\delta x^i} = \partial_i - N_i^a(u)\, \partial_a, \frac{\partial}{\partial y^a} \right), \tag{1.4}$$

or, in brief,$\delta_\alpha = (\delta_i, \partial_a)$, and it dual la–basis

$$\delta u^\alpha = \left(\delta x^i = dx^i, \delta y^a + N_i^a(u)\, dx^i \right), \tag{1.5}$$

or, in brief, $\delta\ ^\alpha = \left(d^i, \delta^a \right)$.

The **nonholonomic coefficients** $\mathbf{w} = \{ w_{\beta\gamma}^\alpha(u) \}$ of la–frames are defined as

$$[\delta_\alpha, \delta_\beta] = \delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = w_{\beta\gamma}^\alpha(u)\, \delta_\alpha. \tag{1.6}$$

The **algebra of tensorial distinguished fields** $DT(\mathcal{E})$ (d–fields, d–tensors, d–objects) on $\mathcal{E}$ is introduced as the tensor algebra $\mathcal{T} = \{ \mathcal{T}_{qs}^{pr} \}$ of the v–bundle $\mathcal{E}_{(d)}$, $p_d : \mathcal{H}E \oplus VE \to E$. An element $\mathbf{t} \in \mathcal{T}_{qs}^{pr}$, d–tensor field of type $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ can be written in local form as

$$\mathbf{t} = t_{j_1 \ldots j_q b_1 \ldots b_r}^{i_1 \ldots i_p a_1 \ldots a_r}(u)\, \delta_{i_1} \otimes \ldots \otimes \delta_{i_p} \otimes \partial_{a_1} \otimes \ldots \otimes \partial_{a_r} \otimes d^{j_1} \otimes \ldots \otimes d^{j_q} \otimes \delta^{b_1} \ldots \otimes \delta^{b_r}.$$

We shall respectively use denotations $\mathcal{X}(E)$ (or $\mathcal{X}(M)$), $\Lambda^p(\mathcal{E})$ (or $\Lambda^p(M)$) and $\mathcal{F}(E)$ (or $\mathcal{F}(M)$) for the module of d–vector fields on $\mathcal{E}$ (or M), the exterior algebra of p–forms on $\mathcal{E}$ (or M) and the set of real functions on $\mathcal{E}$ (or M).

In general, d-objects on $\mathcal{E}$ are introduced as geometric objects with various group and coordinate transforms coordinated with the N-connection structure on $\mathcal{E}$. For example, a d-connection D on $\mathcal{E}$ is defined as a linear connection D on E conserving under a parallelism the global decomposition (1.3) into horizontal and vertical subbundles of $\mathcal{T}E$.

A N-connection in $\mathcal{E}$ induces a corresponding decomposition of d–tensors into sums of horizontal and vertical parts, for example, for every d–vector $X \in \mathcal{X}(E)$ and 1-form $\widetilde{X} \in \Lambda^1(\mathcal{E})$ we have respectively

$$X = hX + vX \quad \text{and} \quad \widetilde{X} = h\widetilde{X} + v\widetilde{X} \ . \tag{1.7}$$

In consequence, we can associate to every d-covariant derivation along the d-vector (1.7), $D_X = X \circ D$, two new operators of h- and v-covariant derivations defined respectively as

$$D_X^{(h)}Y = D_{hX}Y \quad \text{and} \quad D_X^{(v)}Y = D_{vX}Y, \forall Y \in \mathcal{X}(E),$$

for which the following conditions hold:

$$D_X Y = D_X^{(h)}Y + D_X^{(v)}Y, \tag{1.8}$$

$$D_X^{(h)}f = (hX)f \quad \text{and} \quad D_X^{(v)}f = (vX)f, \quad X, Y \in \mathcal{X}(E), f \in \mathcal{F}(M).$$

We define a **metric structure G** in the total space E of v-bundle $\mathcal{E} = (E, p, M)$ over a connected and paracompact base M as a symmetric covariant tensor field of type $(0,2)$, $G_{\alpha\beta},$ being nondegenerate and of constant signature on E.

Definition 1.2. *Nonlinear connection* $\mathbf{N}$ *and metric* $\mathbf{G}$ *structures on* $\mathcal{E}$ *are mutually compatible it there are satisfied the conditions:*

$$\mathbf{G}(\delta_i, \partial_a) = 0, \ \textit{or equivalently,}\ G_{ia}(u) - N_i^b(u)\, h_{ab}(u) = 0, \tag{1.9}$$

where $h_{ab} = \mathbf{G}(\partial_a, \partial_b)$ *and* $G_{ia} = \mathbf{G}(\partial_i, \partial_a)$.

From (1.9) one follows

$$N_i^b(u) = h^{ab}(u)\, G_{ia}(u) \tag{1.10}$$

(the matrix h^{ab} is inverse to h_{ab}). In consequence one obtains the following decomposition of metric :

$$\mathbf{G}(X,Y) = h\mathbf{G}(X,Y) + v\mathbf{G}(X,Y), \tag{1.11}$$

where the d–tensor $h\mathbf{G}(X,Y) = G(hX, hY)$ is of type $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ and the d–tensor $v\mathbf{G}(X,Y) = \mathbf{G}(vX, vY)$ is of type With respect to la–basis (1.5) the d–metric (1.11) is written as

$$\mathbf{G} = g_{\alpha\beta}(u)\, \delta^\alpha \otimes \delta^\beta = g_{ij}(u)\, d^i \otimes d^j + h_{ab}(u)\, \delta^a \otimes \delta^b, \tag{1.12}$$

where $g_{ij} = \mathbf{G}(\delta_i, \delta_j)$.

A metric structure of type (1.11) (equivalently, of type (1.12)) or a metric on E with components satisfying constraints (1.9), equivalently (1.10)) defines an adapted to the given N-connection inner (d-scalar) product on the tangent bundle $\mathcal{T}E$.

Definition 1.3. *We shall say that a d-connection $\widehat{D}_X$ is compatible with the d-scalar product on $\mathcal{T}E$ (i.e. is a standard d-connection) if*

$$\widehat{D}_X\left(\mathbf{X}\cdot Y\right)=\left(\widehat{D}_X\mathbf{Y}\right)\cdot\mathbf{Z}+Y\cdot\left(\widehat{D}_X\mathbf{Z}\right),\forall\mathbf{X},\mathbf{Y},\mathbf{Z}\in\mathcal{X}\left(E\right). \tag{1.13}$$

An arbitrary d-connection D_X differs from the standard one $\widehat{D}_X$ by an operator $\widehat{P}_X\left(u\right)=\{X^\alpha\widehat{P}^\gamma_{\alpha\beta}\left(u\right)\}$, called the deformation d-tensor with respect to $\widehat{D}_X$, which is just a d-linear transform of $\mathcal{E}_u$, $\forall u\in\mathcal{E}$. The explicit form of $\widehat{P}_X$ can be found by using the corresponding axiom defining linear connections [Luehr and Rosenbaum 1974]

$$\left(D_X-\widehat{D}_X\right)fZ=f\left(D_X-\widehat{D}_X\right)Z,$$

written with respect to la–bases (1.4) and (1.5). From the last expression we obtain

$$\widehat{P}_X\left(u\right)=\left[(D_X-\widehat{D}_X)\delta_\alpha\left(u\right)\right]\delta^\alpha\left(u\right), \tag{1.14}$$

therefore

$$D_XZ=\widehat{D}_XZ+\widehat{P}_XZ. \tag{1.15}$$

Definition 1.4. *A d–connection D_X is* **metric (***or* **compatible** *with metric* $\mathbf{G}$*) on $\mathcal{E}$ if*

$$D_X\mathbf{G}=0,\forall X\in\mathcal{X}\left(E\right). \tag{1.16}$$

If there is a complex structure ${J_\alpha}^\beta$, $J\dot{J}=-I$, being compatible with a metric of type (1.12) and a d–connection D on tangent bundle TM, when conditions

$${J_\alpha}^\beta{J_\gamma}^\delta G_{\beta\delta}=G_{\alpha\gamma}\quad\text{and}\quad D_\alpha J^\gamma_{\ \beta}=0$$

are satisfied, one considers that on TM it is defined an almost Hermitian model, H^{2n}-model, of a generalized Lagrange space, GL-space [Miron and Anastasiei 1987, 1994].

Locally adapted components $\Gamma^\alpha_{\beta\gamma}$ of a d-connection $D_\alpha=(\delta_\alpha\circ D)$ are defined by the equations

$$D_\alpha\delta_\beta=\Gamma^\gamma_{\ \alpha\beta}\delta_\gamma,$$

from which one immediately follows

$$\Gamma^\gamma_{\ \alpha\beta}\left(u\right)=\left(D_\alpha\delta_\beta\right)\circ\delta^\gamma. \tag{1.17}$$

The operations of h- and v-covariant derivations, $D^{(h)}_k=\{L^i_{jk},L^a_{bk}\}$ and $D^{(v)}_c=\{C^i_{jk},C^a_{bc}\}$ (see (1.8)), are introduced as corresponding h- and v-parametrizations of (1.17):

$$L^i_{jk}=\left(D_k\delta_j\right)\circ d^i,\quad L^a_{bk}=\left(D_k\partial_b\right)\circ\delta^a \tag{1.18}$$

and

$$C^i_{jc} = (D_c\delta_j) \circ d^i, \quad C^a_{bc} = (D_c\partial_b) \circ \delta^a. \tag{1.19}$$

A set of components (1.18) and (1.19), $D\Gamma = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}\right)$, completely defines the local action of a d–connection D in $\mathcal{E}$. For instance, taken a d–tensor field of type $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ $\mathbf{t} = t^{ia}_{jb}\delta_i \otimes \partial_a \otimes \partial^j \otimes \delta^b$, and a d-vector $\mathbf{X} = X^i\delta_i + X^a\partial_a$ we have

$$D_X\mathbf{t} = D^{(h)}_X\mathbf{t} + D^{(v)}_X\mathbf{t} = \left(X^k t^{ia}_{jb|k} + X^c t^{ia}_{jb\perp c}\right)\delta_i \otimes \partial_a \otimes d^j \otimes \delta^b,$$

where the h-covariant derivative is written as

$$t^{ia}_{jb|k} = \frac{\delta t^{ia}_{jb}}{\delta x^k} + L^i_{hk}t^{ha}_{jb} + L^a_{ck}t^{ic}_{jb} - L^h_{jk}t^{ia}_{hb} - L^c_{bk}t^{ia}_{jc}$$

and the v-covariant derivative is written as

$$t^{ia}_{jb\perp c} = \frac{\partial t^{ia}_{jb}}{\partial y^c} + C^i_{hc}t^{ha}_{jb} + C^a_{dc}t^{id}_{jb} - C^h_{jc}t^{ia}_{hb} - C^d_{bc}t^{ia}_{jd}.$$

For a scalar function $f \in \mathcal{F}(E)$ we have

$$D^{(h)}_k = \frac{\delta f}{\delta x^k} = \frac{\partial f}{\partial x^k} - N^a_k\frac{\partial f}{\partial y^a} \text{ and } D^{(v)}_c f = \frac{\partial f}{\partial y^c}.$$

We emphasize that the geometry of connections in a v-bundle $\mathcal{E}$ is very reach. If a triple of fundamental geometric objects $\left(N^a_i(u), \Gamma^\alpha_{\beta\gamma}(u), G_{\alpha\beta}(u)\right)$ is fixed on $\mathcal{E}$, really, a multiconnection structure (with corresponding different rules of covariant derivation, which are, or not, mutually compatible and with the same, or not, induced d-scalar products in $\mathcal{T}E$) is defined on this v-bundle. For instance, we enumerate some of connections and covariant derivations which can present interest in investigation of locally anisotropic gravitational and matter field interactions:

1 . Every N-connection in $\mathcal{E}$, with coefficients $N^a_i(x,y)$ being differentiable on y-variables, induces a structure of linear connection $\widetilde{N}^\alpha_{\beta\gamma}$, where $\widetilde{N}^a_{bi} = \frac{\partial N^a_i}{\partial y^b}$ and $\widetilde{N}^a_{bc}(x,y) = 0$. For some $Y(u) = Y^i(u)\partial_i + Y^a(u)\partial_a$ and $B(u) = B^a(u)\partial_a$ one writes

$$D^{(\widetilde{N})}_Y B = \left[Y^i\left(\frac{\partial B^a}{\partial x^i} + \widetilde{N}^a_{bi}B^b\right) + Y^b\frac{\partial B^a}{\partial y^b}\right]\frac{\partial}{\partial y^a}.$$

2 . The d-connection of Berwald type [Berwald 1926]

$$\Gamma^{(B)\alpha}_{\beta\gamma} = \left(L^i_{jk}, \frac{\partial N^a_k}{\partial y^b}, 0, C^a_{bc}\right), \tag{1.20}$$

where

$$L^i_{.jk}(x,y) = \frac{1}{2}g^{ir}\left(\frac{\delta g_{jk}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r}\right), \tag{1.21}$$

$$C^a_{.bc}(x,y) = \frac{1}{2}h^{ad}\left(\frac{\partial h_{bd}}{\partial y^c} + \frac{\partial h_{cd}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^d}\right),$$

which is hv-metric, i.e. $D_k^{(B)} g_{ij} = 0$ and $D_c^{(B)} h_{ab} = 0$.

3 . The canonical d-connection $\Gamma^{(c)}$ associated to a metric $\mathbf{G}$ of type (1.12) $\Gamma^{(c)\alpha}_{\beta\gamma} = \left(L^{(c)i}_{jk}, L^{(c)a}_{bk}, C^{(c)i}_{jc}, C^{(c)a}_{bc}\right)$, with coefficients

(1.22) $L^{(c)i}_{jk} = L^{i}_{.jk}, C^{(c)a}_{bc} = C^{a}_{.bc}$ (see (1.21)

$$L^{(c)a}_{bi} = \widetilde{N}^{a}_{bi} + \frac{1}{2} h^{ac} \left(\frac{\delta h_{bc}}{\delta x^i} - \widetilde{N}^{d}_{bi} h_{dc} - \widetilde{N}^{d}_{ci} h_{db} \right),\ C^{(c)i}_{jc} = \frac{1}{2} g^{ik} \frac{\partial g_{jk}}{\partial y^c}.$$

This is a metric d–connection which satisfies conditions

$$D_k^{(c)} g_{ij} = 0, D_c^{(c)} g_{ij} = 0, D_k^{(c)} h_{ab} = 0, D_c^{(c)} h_{ab} = 0.$$

4 . We can consider N–adapted Christoffel d–symbols

$$\widetilde{\Gamma}^{\alpha}_{\beta\gamma} = \frac{1}{2} G^{\alpha\tau} \left(\delta_\gamma G_{\tau\beta} + \delta_\beta G_{\tau\gamma} - \delta G_{\beta\gamma} \right), \tag{1.23}$$

which have the components of d-connection $\widetilde{\Gamma}^{\alpha}_{\beta\gamma} = \left(L^{i}_{jk}, 0, 0, C^{a}_{bc}\right)$, with L^{i}_{jk} and C^{a}_{bc} as in (1.21) if $G_{\alpha\beta}$ is taken in the form (1.12).

Arbitrary linear connections on a v–bundle $\mathcal{E}$ can be also characterized by theirs deformation tensors (see (1.15)) with respect, for instance, to d-connection (1.23):

$$\Gamma^{(B)\alpha}_{\beta\gamma} = \widetilde{\Gamma}^{\alpha}_{\beta\gamma} + P^{(B)\alpha}_{\beta\gamma}, \Gamma^{(c)\alpha}_{\beta\gamma} = \widetilde{\Gamma}^{\alpha}_{\beta\gamma} + P^{(c)\alpha}_{\beta\gamma}$$

or, in general,

$$\Gamma^{\alpha}_{\beta\gamma} = \widetilde{\Gamma}^{\alpha}_{\beta\gamma} + P^{\alpha}_{\beta\gamma},$$

where $P^{(B)\alpha}_{\beta\gamma}, P^{(c)\alpha}_{\beta\gamma}$ and $P^{\alpha}_{\beta\gamma}$ are respectively the deformation d-tensors of d-connections (1.20), (1.22), or of a general one.

I.1.2 Torsions and curvatures of nonlinear and distinguished connections.

The curvature Ω, of a nonlinear connection $\mathbf{N}$ in a v–bundle $\mathcal{E}$ can be defined as the Nijenhuis tensor field $N_v(X,Y)$ associated to $\mathbf{N}$:

$$\Omega = N_v = [vX, vY] + v[X,Y] - v[vX,Y] - v[X,vY], X, Y \in \mathcal{X}(E).$$

In local form one has

$$\Omega = \frac{1}{2} \Omega^{a}_{ij} d^i \bigwedge d^j \otimes \partial_a,$$

where

$$\Omega^{a}_{ij} = \frac{\partial N^a_i}{\partial x^j} - \frac{\partial N^a_j}{\partial x^i} + N^b_i \widetilde{N}^a_{bj} - N^b_j \widetilde{N}^a_{bi}. \tag{1.24}$$

The torsion $\mathbf{T}$ of a d–connection $\mathbf{D}$ in $\mathcal{E}$ is defined by the equation

$$\mathbf{T}(X,Y) = XY_{\circ}^{\circ}T \doteq D_X\mathbf{Y} - D_Y\mathbf{X} - [X,Y]. \tag{1.25}$$

One holds the following h- and v–decompositions

$$\mathbf{T}(X,Y) = T(hX,hY) + T(hX,vY) + T(vX,hY) + T(vX,vY). \tag{1.26}$$

We consider the projections:$h\mathbf{T}(X,Y), vT(hX,hY), hT(hX,hY), ...$ and say that, for instance, $h\mathbf{T}(hX,hY)$ is the h(hh)-torsion of $\mathbf{D}$, $v\mathbf{T}(hX,hY)$ is the v(hh)-torsion of $\mathbf{D}$ and so on.

The torsion (1.25) is locally determined by five d–tensor fields, torsions, defined as

$$\begin{aligned} & T^i_{jk} = h\mathbf{T}(\delta_k,\delta_j)\cdot d^i, \quad T^a_{jk} = v\mathbf{T}(\delta_k,\delta_j)\cdot \delta^a, \\ & P^i_{jb} = h\mathbf{T}(\partial_b,\delta_j)\cdot d^i, \quad P^a_{jb} = v\mathbf{T}(\partial_b,\delta_j)\cdot \delta^a, \quad S^a_{bc} = v\mathbf{T}(\partial_c,\partial_b)\cdot \delta^a. \end{aligned} \tag{1.27}$$

Using formulas (1.4),(1.5),(1.24) and (1.25) we can computer in explicit form the components of torsions (1.26) for a d–connection of type (1.18) and (1.19):

(1.28)
$$\begin{gathered} T^i_{.jk} = T^i_{jk} = L^i_{jk} - L^i_{kj},\ T^i_{ja} = C^i_{.ja},\ T^i_{aj} = -C^i_{ja},\ T^a_{.bc} = S^a_{.bc} = C^a_{bc} - C^a_{cb}, \\ T^i_{.ja} = 0,\ T^a_{.ij} = \frac{\delta N^a_i}{\delta x^j} - \frac{\delta N^a_j}{\delta x^i},\ T^a_{.bi} = P^a_{.bi} = \frac{\partial N^a_i}{\partial y^b} - L^a_{.bj},\ T^a_{.ib} = -P^a_{.bi}. \end{gathered}$$

The curvature $\mathbf{R}$ of a d–connection in $\mathcal{E}$ is defined by the equation

$$\mathbf{R}(X,Y)Z = XY_{\bullet}^{\bullet}R \bullet Z = D_X D_Y \mathbf{Z} - D_Y D_X \mathbf{Z} - D_{[X,Y]}\mathbf{Z}. \tag{1.29}$$

One holds the next properties for the h- and v-decompositions of curvature:

$$\begin{gathered} v\mathbf{R}(X,Y)hZ = 0,\ h\mathbf{R}(X,Y)vZ = 0, \\ \mathbf{R}(X,Y)Z = hR(X,Y)hZ + vR(X,Y)vZ. \end{gathered} \tag{1.30}$$

From (1.29) and the equation $\mathbf{R}(X,Y) = -R(Y,X)$ we get that the curvature of a d-connection $\mathbf{D}$ in $\mathcal{E}$ is completely determined by the following six d–tensor fields:

(1.31)
$$\begin{gathered} R^{.i}_{h.jk} = d^i\cdot\mathbf{R}(\delta_k,\delta_j)\delta_h,\ R^{.a}_{b.jk} = \delta^a\cdot\mathbf{R}(\delta_k,\delta_j)\partial_b,\ P^{.i}_{j.kc} = d^i\cdot\mathbf{R}(\partial_c,\partial_k)\delta_j, \\ P^{.a}_{b.kc} = \delta^a\cdot\mathbf{R}(\partial_c,\partial_k)\partial_b,\ S^{.i}_{j.bc} = d^i\cdot\mathbf{R}(\partial_c,\partial_b)\delta_j,\ S^{.a}_{b.cd} = \delta^a\cdot\mathbf{R}(\partial_d,\partial_c)\partial_b. \end{gathered}$$

By a direct computation, using (1.4),(1.5),(1.18),(1.19) and (1.31) we get

(1.32)
$$R_{h.jk}^{.i} = \frac{\delta L_{.hj}^{i}}{\delta x^{h}} - \frac{\delta L_{.hk}^{i}}{\delta x^{j}} + L_{.hj}^{m} L_{mk}^{i} - L_{.hk}^{m} L_{mj}^{i} + C_{.ha}^{i} R_{.jk}^{a},$$
$$R_{b.jk}^{.a} = \frac{\delta L_{.bj}^{a}}{\delta x^{k}} - \frac{\delta L_{.bk}^{a}}{\delta x^{j}} + L_{.bj}^{c} L_{.ck}^{a} - L_{.bk}^{c} L_{.cj}^{a} + C_{.bc}^{a} R_{.jk}^{c},$$
$$P_{j.ka}^{.i} = \frac{\partial L_{.jk}^{i}}{\partial y^{k}} - \left(\frac{\partial C_{.ja}^{i}}{\partial x^{k}} + L_{.lk}^{i} C_{.ja}^{l} - L_{.jk}^{l} C_{.la}^{i} - L_{.ak}^{c} C_{.jc}^{i} \right) + C_{.jb}^{i} P_{.ka}^{b},$$
$$P_{b.ka}^{.c} = \frac{\partial L_{.bk}^{c}}{\partial y^{a}} - \left(\frac{\partial C_{.ba}^{c}}{\partial x^{k}} + L_{.dk}^{c} C_{.ba}^{d} - L_{.bk}^{d} C_{.da}^{c} - L_{.ak}^{d} C_{.bd}^{c} \right) + C_{.bd}^{c} P_{.ka}^{d},$$
$$S_{j.bc}^{.i} = \frac{\partial C_{.jb}^{i}}{\partial y^{c}} - \frac{\partial C_{.jc}^{i}}{\partial y^{b}} + C_{.jb}^{h} C_{.hc}^{i} - C_{.jc}^{h} C_{hb}^{i},$$
$$S_{b.cd}^{.a} = \frac{\partial C_{.bc}^{a}}{\partial y^{d}} - \frac{\partial C_{.bd}^{a}}{\partial y^{c}} + C_{.bc}^{e} C_{.ed}^{a} - C_{.bd}^{e} C_{.ec}^{a}.$$

We note that torsions (1.28) and curvatures (1.32) can be computed by particular cases of d-connections when d-connections (1.20), (1.22) or (1.24) are used instead of (1.18) and (1.19).

The components of the Ricci d-tensor

$$R_{\alpha\beta} = R_{\alpha.\beta\tau}^{.\tau}$$

with respect to locally adapted frame (1.5) are as follows:

$$R_{ij} = R_{i.jk}^{.k}, \quad R_{ia} = -{}^{2}P_{ia} = -P_{i.ka}^{.k}, \quad R_{ai} = {}^{1}P_{ai} = P_{a.ib}^{.b}, \quad R_{ab} = S_{a.bc}^{.c}. \tag{1.33}$$

We point out that because, in general, ${}^{1}P_{ai} \neq {}^{2}P_{ia}$ the Ricci d–tensor is non symmetric.

Having defined a d-metric of type (1.12) in $\mathcal{E}$ we can introduce the scalar curvature of d–connection $\mathbf{D}$:

$$\overleftarrow{R} = G^{\alpha\beta} R_{\alpha\beta} = R + S, \tag{1.34}$$

where $R = g^{ij} R_{ij}$ and $S = h^{ab} S_{ab}$.

For our further considerations it will be also useful to consider an alternative way of definition torsion (1.25) and curvature (1.29) by using the commutator

$$\Delta_{\alpha\beta} \doteq \nabla_{\alpha}\nabla_{\beta} - \nabla_{\beta}\nabla_{\alpha} = 2\nabla_{[\alpha}\nabla_{\beta]}. \tag{1.35}$$

For components (1.28) of d-torsion we have

$$\Delta_{\alpha\beta} f = T_{.\alpha\beta}^{\gamma} \nabla_{\gamma} f \tag{1.36}$$

for every scalar function f on $\mathcal{E}$. Curvature can be introduced as an operator acting on arbitrary d-vector V^δ :

$$(\Delta_{\alpha\beta} - T^{\gamma}_{.\alpha\beta}\nabla_\gamma)V^\delta = R^{\cdot\delta}_{\gamma.\alpha\beta}V^\gamma \tag{1.37}$$

(we note that in this Chapter we shall follow conventions of [Miron and Anastasiei 1987, 1994] on d–tensors; we can obtain corresponding abstract index formulas from [Penrose and Rindler 1984, 1986] just for a trivial N–connection structure and by changing denotations for components of torsion and curvature in this manner: $T^{\gamma}_{.\alpha\beta} \to T_{\alpha\beta}{}^{\gamma}$ and $R^{\cdot\delta}_{\gamma.\alpha\beta} \to R_{\alpha\beta\gamma}{}^{\delta}$).

Here we also note that torsion and curvature of a d-connection on $\mathcal{E}$ satisfy generalized for la–spaces Ricci and Bianchi identities [Miron and Anastasiei 1987, 1994] which in terms of components (1.36) and (1.37) are written respectively as

$$R^{\cdot\delta}_{[\gamma.\alpha\beta]} + \nabla_{[\alpha}T^{\delta}_{.\beta\gamma]} + T^{\nu}_{.[\alpha\beta}T^{\delta}_{.\gamma]\nu} = 0 \tag{1.38}$$

and

$$\nabla_{[\alpha}R^{\cdot\sigma}_{|\nu|\beta\gamma]} + T^{\delta}_{\cdot[\alpha\beta}R^{\cdot\sigma}_{|\nu|.\gamma]\delta} = 0. \tag{1.39}$$

Identities (1.38) and (1.39) can be proved by straightforward calculations.

We can also consider a la–generalization of the so–called conformal Weyl tensor (see, for instance, [Penrose and Rindler 1984]):

$$C^{\gamma\delta}{}_{\alpha\beta} = R^{\gamma\delta}{}_{\alpha\beta} - \frac{4}{n+m-2}R^{[\gamma}{}_{[\alpha}\ \delta^{\delta]}{}_{\beta]} + \frac{2}{(n+m-1)(n+m-2)}\overleftarrow{R}\ \delta^{[\gamma}{}_{[\alpha}\ \delta^{\delta]}{}_{\beta]}. \tag{1.40}$$

This d–tensor is conformally invariant on la-spaces provided with d-connection generated by d–metric structures.

I.1.3 Field equations for locally anisotropic gravity.

The Einstein equations and the problem of conservation laws on v–bundles provided with N–connection structures are considered in [Miron and Anastasiei 1994]. In work [Vacaru and Goncharenko 1995] it was proved that the la–gravity can be formulated in a gauge like manner and the conditions when the Einstein la–gravitational field equations are equivalent to a corresponding form of Yang-Mills equations where analized. In this subsection we shall express the la–gravitational field equations in a form more convenient for theirs equivalent reformulation in la–spinor variables.

We define d-tensor $\Phi_{\alpha\beta}$ as to satisfy conditions

$$-2\Phi_{\alpha\beta} \doteq R_{\alpha\beta} - \frac{1}{n+m}\overleftarrow{R}g_{\alpha\beta} \tag{1.41}$$

which is the torsionless part of the Ricci tensor for locally isotropic spaces [Penrose and Rindler 1984], i.e. $\Phi_{\alpha}{}^{\alpha} \doteq 0$. The Einstein equations on la–spaces

$$\overleftarrow{G}_{\alpha\beta} + \lambda g_{\alpha\beta} = \kappa E_{\alpha\beta}, \tag{1.42}$$

where

$$\overleftarrow{G}_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}\overleftarrow{R}\, g_{\alpha\beta} \tag{1.43}$$

is the Einstein d-tensor, λ and κ are correspondingly the cosmological and gravitational constants and by $E_{\alpha\beta}$ is denoted the locally anisotropic energy–momentum d–tensor [Miron and Anastasiei 1987, 1994], can be rewritten in equivalent form:

$$\Phi_{\alpha\beta} = -\frac{\kappa}{2}(E_{\alpha\beta} - \frac{1}{n+m}E_{\tau}^{\ \tau}\ g_{\alpha\beta}). \tag{1.44}$$

Because la–spaces generally have nonzero torsions we shall add to (1.44) (equivalently to (1.42)) a system of algebraic d–field equations with the source $S^{\alpha}{}_{\beta\gamma}$ being the locally anisotropic spin density of matter (if we consider a variant of locally anisotropic Einstein–Cartan theory):

$$T^{\gamma}{}_{\alpha\beta} + 2\delta^{\gamma}{}_{[\alpha}T^{\delta}{}_{\beta]\delta} = \kappa S^{\gamma}{}_{\alpha\beta.} \tag{1.45}$$

From (1.38) and (1.45) one follows the conservation law of locally anisotropic spin matter:

$$\nabla_{\gamma}S^{\gamma}{}_{\alpha\beta} - T^{\delta}{}_{\delta\gamma}S^{\gamma}{}_{\alpha\beta} = E_{\beta\alpha} - E_{\alpha\beta}.$$

Finally, in this section, we remark that all presented geometric constructions contain those elaborated for generalized Lagrange spaces [Miron and Anastasiei 1987, 1994] (for which a tangent bundle TM is considered instead of a v-bundle $\mathcal{E}$). Here we note that the Lagrange (Finsler) geometry is characterized by a metric of type (1.12) with components parametized as $g_{ij} = \frac{1}{2}\frac{\partial^2\mathcal{L}}{\partial y^i \partial y^j}$ $\left(g_{ij} = \frac{1}{2}\frac{\partial^2\Lambda^2}{\partial y^i\partial y^j}\right)$ and $h_{ij} = g_{ij}$, where $\mathcal{L} = L\ (x,y)$ $(\Lambda = \Lambda\,(x,y))$ is a Lagrangian (Finsler metric) on TM (see details in [Miron and Anastasiei 1994], [Matsumoto 1986] and [Bejancu 1990]).

I.2 Distinguished Clifford Algebras

The typical fiber of a v–bundle ξ_d , $\pi_d :\ HE \oplus VE \to E$ is a d-vector space, $\mathcal{F} = h\mathcal{F} \oplus v\mathcal{F}$, split into horizontal $h\mathcal{F}$ and vertical $v\mathcal{F}$ subspaces, with metric $G(g,h)$ induced by v-bundle metric (1.12). Clifford algebras (see, for example, [Karoubi 1978] and [Penrose and Rindler 1986]) formulated for d–vector spaces will be called Clifford d–algebras [Vacaru 1996]. In this section we shall consider the main properties of Clifford d–algebras. The proof of theorems will be based on the technique developed in [Karoubi 1978] correspondingly adapted to the distinguished character of spaces in consideration.

Let k be a number field (for our purposes $k = \mathbb{R}$ or $k = \mathbb{C}$, $\mathbb{R}$ and $\mathbb{C}$ are respectively real and complex number fields) and define $\mathcal{F}$ as a d–vector space on k provided with a nondegenerate symmetric quadratic form (metric) G. Let C be an algebra on k (not necessarily commutative) and $j\ :\ \mathcal{F} \to C$ a homomorphism of underlying vector spaces such that $j(u)^2 =\ G(u)\cdot 1$ (1 is the unity in algebra C

and d-vector $u \in \mathcal{F}$). We are interested in definition of the pair (C, j) satisfying the next universitality conditions. For every k-algebra A and arbitrary homomorphism $\varphi : \mathcal{F} \to A$ of the underlying d-vector spaces, such that $(\varphi(u))^2 \to G(u) \cdot 1$, there is a unique homomorphism of algebras $\psi : C \to A$ transforming the diagram

$$\begin{array}{ccc} \mathcal{F} & \stackrel{j}{\hookrightarrow} & C \\ \varphi\downarrow & & \downarrow\tau \\ A & \Longleftrightarrow & A \end{array}$$

into a commutative one. The algebra solving this problem will be denoted as $C(\mathcal{F}, A)$ [equivalently as $C(G)$ or $C(\mathcal{F})$ and called as Clifford d–algebra associated with pair $(\mathcal{F}, G)$.

Theorem 1.1. *The above-presented diagram has a unique solution (C, j) up to isomorphism.*

Proof: (We adapt for d-algebras that of [Karoubi 1978], p. 127.) For a universal problem the uniqueness is obvious if we prove the existence of solution $C(G)$. To do this we use tensor algebra $\mathcal{L}^{(F)} = \oplus \mathcal{L}^{pr}_{qs}(\mathcal{F}) = \oplus_{i=0}^{\infty} T^i(\mathcal{F})$, where $T^0(\mathcal{F}) = k$ and $T^i(\mathcal{F}) = k$ and $T^i(\mathcal{F}) = \mathcal{F} \otimes ... \otimes \mathcal{F}$ for $i > 0$. Let $I(G)$ be the bilateral ideal generated by elements of form $\epsilon(u) = u \otimes u - G(u) \cdot 1$ where $u \in \mathcal{F}$ and 1 is the unity element of algebra $\mathcal{L}(\mathcal{F})$. Every element from $I(G)$ can be written as $\sum_i \lambda_i \epsilon(u_i) \mu_i$, where $\lambda_i, \mu_i \in \mathcal{L}(\mathcal{F})$ and $u_i \in \mathcal{F}$. Let $C(G) = \mathcal{L}(\mathcal{F})/I(G)$ and define $j : \mathcal{F} \to C(G)$ as the composition of monomorphism $i : \mathcal{F} \to L^1(\mathcal{F}) \subset \mathcal{L}(\mathcal{F})$ and projection $p : \mathcal{L}(\mathcal{F}) \to C(G)$. In this case pair $(C(G), j)$ is the solution of our problem. From the general properties of tensor algebras the homomorphism $\varphi : \mathcal{F} \to A$ can be extended to $\mathcal{L}(\mathcal{F})$, i.e., the diagram

$$\begin{array}{ccc} \mathcal{F} & \stackrel{j}{\longrightarrow} & \mathcal{L}(\mathcal{F}) \\ \varphi\downarrow & & \downarrow\rho \\ A & \Longleftrightarrow & A \end{array}$$

is commutative, where ρ is a monomorphism of algebras. Because $(\varphi(u))^2 = G(u) \cdot 1$, then ρ vanishes on ideal $I(G)$ and in this case the necessary homomorphism τ is defined. As a consequence of uniqueness of ρ, the homomorphism τ is unique. ■

Tensor d-algebra $\mathcal{L}(\mathcal{F})$ can be considered as a $\mathbb{Z}/2$ graded algebra. Really, let us introduce $\mathcal{L}^{(0)}(\mathcal{F}) = \sum_{i=1}^{\infty} T^{2i}(\mathcal{F})$ and $\mathcal{L}^{(1)}(\mathcal{F}) = \sum_{i=1}^{\infty} T^{2i+1}(\mathcal{F})$. Setting $I^{(\alpha)}(G) = I(G) \cap \mathcal{L}^{(\alpha)}(\mathcal{F})$. Define $C^{(\alpha)}(G)$ as $p\left(\mathcal{L}^{(\alpha)}(\mathcal{F})\right)$, where $p : \mathcal{L}(\mathcal{F}) \to C(G)$ is the canonical projection. Then $C(G) = C^{(0)}(G) \oplus C^{(1)}(G)$ and in consequence we obtain that the Clifford d–algebra is $\mathbb{Z}/2$ graded.

It is obvious that Clifford d–algebra functorially depends on pair $(\mathcal{F}, G)$. If $f : \mathcal{F} \to \mathcal{F}'$ is a homomorphism of k-vector spaces, such that $G'(f(u)) = G(u)$, where G and G' are, respectively, metrics on $\mathcal{F}$ and $\mathcal{F}'$, then f induces an homomorphism of d–algebras

$$C(f) : C(G) \to C(G')$$

with identities $C(\varphi \cdot f) = C(\varphi)\, C(f)$ and $C(Id_{\mathcal{F}}) = Id_{C(\mathcal{F})}$.

If $\mathcal{A}^{\alpha}$ and $\mathcal{B}^{\beta}$ are $\mathbb{Z}/2$–graded d–algebras, then their graded tensorial product $\mathcal{A}^{\alpha} \otimes \mathcal{B}^{\beta}$ is defined as a d-algebra for k-vector d-space $\mathcal{A}^{\alpha} \otimes \mathcal{B}^{\beta}$ with the graded product induced as $(a \otimes b)(c \otimes d) = (-1)^{\alpha\beta} ac \otimes bd$, where $b \in \mathcal{B}^{\alpha}$ and $c \in \mathcal{A}^{\alpha} \quad (\alpha, \beta = 0, 1)$.

Now we reformulate for d–algebras the theorem [Chevalley 1955]:

Theorem 1.2. *The Clifford d-algebra $C(h\mathcal{F} \oplus v\mathcal{F}, g + h)$ is naturally isomorphic to $C(g) \otimes C(h)$.*

Proof. Let $n : h\mathcal{F} \to C(g)$ and $n' : v\mathcal{F} \to C(h)$ be canonical maps and map $m : h\mathcal{F} \oplus v\mathcal{F} \to C(g) \otimes C(h)$ is defined as $m(x,y) = n(x) \otimes 1 + 1 \otimes n'(y)$, $x \in h\mathcal{F}, y \in v\mathcal{F}$. We have $(m(x,y))^2 = \left[(n(x))^2 + (n'(y))^2\right] \cdot 1 = [g(x) + h(y)]$. Taking into account the universality property of Clifford d–algebras we conclude that m induces the homomorphism

$$\zeta : C(h\mathcal{F} \oplus v\mathcal{F}, g + h) \to C(h\mathcal{F}, g)\,\widehat{\otimes} C(v\mathcal{F}, h).$$

We also can define a homomorphism

$$\upsilon : C(h\mathcal{F}, g)\,\widehat{\otimes} C(v\mathcal{F}, h) \to C(h\mathcal{F} \oplus v\mathcal{F}, g + h)$$

by using formula $\upsilon(x \otimes y) = \delta(x)\,\delta'(y)$, where homomorphysms δ and δ' are, respectively, induced by imbeddings of $h\mathcal{F}$ and $v\mathcal{F}$ into $h\mathcal{F} \oplus v\mathcal{F}$:

$$\delta : C(h\mathcal{F}, g) \to C(h\mathcal{F} \oplus v\mathcal{F}, g + h),$$

$$\delta' : C(v\mathcal{F}, h) \to C(h\mathcal{F} \oplus v\mathcal{F}, g + h).$$

Because $x \in C^{(\alpha)}(g)$ and $y \in C^{(\alpha)}(g)$, $\delta(x)\,\delta'(y) = (-1)^{(\alpha)}\,\delta'(y)\,\delta(x)$.

Superpositions of homomorphisms ζ and υ lead to identities

$$\upsilon\zeta = Id_{C(h\mathcal{F},g)\widehat{\otimes}C(v\mathcal{F},h)}, \tag{1.46}$$

$$\zeta\upsilon = Id_{C(h\mathcal{F},g)\widehat{\otimes}C(v\mathcal{F},h)}.$$

Really, d–algebra $C(h\mathcal{F} \oplus v\mathcal{F}, g + h)$ is generated by elements of type $m(x,y)$. Calculating

$$\upsilon\zeta(m(x,y)) = \upsilon(n(x) \otimes 1 + 1 \otimes n'(y)) = \delta(n(x))\,\delta(n'(y)) =$$

$$m(x,0) + m(0,y) = m(x,y),$$

we prove the first identity in (1.46).

On the other hand, d-algebra $C(h\mathcal{F}, g)\,\widehat{\otimes} C(v\mathcal{F}, h)$ is generated by elements of type $n(x) \otimes 1$ and $1 \otimes n'(y)$, we prove the second identity in (1.46).

Following from the above–mentioned properties of homomorphisms ζ and υ we can assert that the natural isomorphism is explicitly constructed.■

In consequence of theorem 1.2 we conclude that all operations with Clifford d–algebras can be reduced to calculations for $C(h\mathcal{F},g)$ and $C(v\mathcal{F},h)$ which are usual Clifford algebras of dimension 2^n and, respectively, 2^m [Atiyah, Bott and Shapiro 1964] and [Karoubi 1978].

Of special interest is the case when $k=\mathcal{R}$ and $\mathcal{F}$ is isomorphic to vector space $\mathbb{R}^{p+q,a+b}$ provided with quadratic form $-x_1^2-...-x_p^2+x_{p+q}^2-y_1^2-...-y_a^2+...+y_{a+b}^2$. In this case, the Clifford algebra, denoted as $\left(C^{p,q},C^{a,b}\right)$, is generated by symbols $e_1^{(x)},e_2^{(x)},...,e_{p+q}^{(x)},e_1^{(y)},e_2^{(y)},...,e_{a+b}^{(y)}$ satisfying properties $\left(e_i\right)^2=-1\ (1\leq i\leq p),\left(e_j\right)^2=-1\ (1\leq j\leq a),\left(e_k\right)^2=1\ (p+1\leq k\leq p+q)$,

$\left(e_j\right)^2=1\ (n+1\leq s\leq a+b),\ e_ie_j=-e_je_i,\ i\neq j$. Explicit calculations of $C^{p,q}$ and $C^{a,b}$ are possible by using isomorphisms [Karoubi 1978] and [Penrose and Rindler 1986]

$$C^{p+n,q+n}\simeq C^{p,q}\otimes M_2(\mathbb{R})\otimes...\otimes M_2(\mathbb{R})\cong C^{p,q}\otimes M_{2^n}(\mathbb{R})\cong M_{2^n}\left(C^{p,q}\right),$$

where $M_s(A)$ denotes the ring of quadratic matrices of order s with coefficients in ring A. Here we write the simplest isomorphisms $C^{1,0}\simeq\mathbb{C},C^{0,1}\simeq\mathbb{R}\oplus\mathbb{R}$, and $C^{2,0}=\mathbb{H}$, where by $\mathbb{H}$ is denoted the body of quaternions. We summarize this calculus as

$$C^{0,0}=\mathbb{R},C^{1,0}=\mathbb{C},C^{0,1}=\mathbb{R}\oplus\mathbb{R},C^{2,0}=\mathbb{H},C^{0,2}=M_2(\mathbb{R}),$$

$$C^{3,0}=\mathbb{H}\oplus\mathbb{H},C^{0,3}=M_2(\mathbb{R}),C^{4,0}=M_2(\mathbb{H}),C^{0,4}=M_2(\mathbb{H}),$$

$$C^{5,0}=M_4(\mathbb{C}),\ C^{0,5}=M_2(\mathbb{H})\oplus M_2(\mathbb{H}),\ C^{6,0}=M_8(\mathbb{R}),\ C^{0,6}=M_4(\mathbb{H}),$$

$$C^{7,0}=M_8(\mathbb{R})\oplus M_8(\mathbb{R}),\ C^{0,7}=M_8(\mathbb{C}),\ C^{8,0}=M_{16}(\mathbb{R}),\ C^{0,8}=M_{16}(\mathbb{R}).$$

One of the most important properties of real algebras $C^{0,p}\ \left(C^{0,a}\right)$ and $C^{p,0}\ \left(C^{a,0}\right)$ is eightfold periodicity of $p(a)$.

Now, we emphasize that H^{2n}-spaces admit locally a structure of Clifford algebra on complex vector spaces. Really, by using almost Hermitian structure $J_\alpha{}^\beta$ and considering complex space $\mathbb{C}^n$ with nondegenarate quadratic form $\sum_{a=1}^n\left|z_a\right|^2,\ z_a\in\mathbb{C}^2$ induced locally by metric (1.12) (rewritten in complex coordinates $z_a=x_a+iy_a$) we define Clifford algebra $\overleftarrow{C}^n=\underbrace{\overleftarrow{C}^1\otimes...\otimes\overleftarrow{C}^1}_n$, where $\overleftarrow{C}^1=\mathbb{C}\otimes_R\mathbb{C}=\mathbb{C}\oplus\mathbb{C}$ or in consequence, $\overleftarrow{C}^n\simeq C^{n,0}\otimes_{\mathbb{R}}\mathbb{C}\approx C^{0,n}\otimes_{\mathbb{R}}\mathbb{C}$. Explicit calculations lead to isomorphisms $\overleftarrow{C}^2=C^{0,2}\otimes_{\mathbb{R}}\mathbb{C}\approx M_2(\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C}\approx M_2\left(\overleftarrow{C}^n\right),\ C^{2p}\approx M_{2^p}(\mathbb{C})$ and $\overleftarrow{C}^{2p+1}\approx M_{2^p}(\mathbb{C})\oplus M_{2^p}(\mathbb{C})$, which show that complex Clifford algebras, defined locally for H^{2n}-spaces, have periodicity 2 on p.

Considerations presented in the proof of theorem 1.2 show that map $j:\mathcal{F}\rightarrow C(\mathcal{F})$ is monomorphic, so we can identify space $\mathcal{F}$ with its image in $C(\mathcal{F},G)$, denoted as $u\rightarrow\overline{u}$, if $u\in C^{(0)}(\mathcal{F},G)\ \left(u\in C^{(1)}(\mathcal{F},G)\right)$; then $u=\overline{u}$ (respectively, $\overline{u}=-u$).

Definition 1.5. *The set of elements* $u \in C(G)^*$, *where* $C(G)^*$ *denotes the multiplicative group of invertible elements of* $C(\mathcal{F}, G)$ *satisfying* $\overline{u}\mathcal{F}u^{-1} \in \mathcal{F}$, *is called the twisted Clifford d–group, denoted as* $\widetilde{\Gamma}(\mathcal{F})$.

Let $\widetilde{\rho} : \widetilde{\Gamma}(\mathcal{F}) \to GL(\mathcal{F})$ be the homorphism given by $u \to \rho\widetilde{u}$, where $\widetilde{\rho}_u(w) = \overline{u}wu^{-1}$. We can verify that $\ker \widetilde{\rho} = \mathbb{R}^*$is a subgroup in $\widetilde{\Gamma}(\mathcal{F})$.

Canonical map $j : \mathcal{F} \to C(\mathcal{F})$ can be interpreted as the linear map $\mathcal{F} \to C(\mathcal{F})^0$ satisfying the universal property of Clifford d-algebras. This leads to a homomorphism of algebras, $C(\mathcal{F}) \to C(\mathcal{F})^t$, considered by an anti–involution of $C(\mathcal{F})$ and denoted as $u \to {}^t u$. More exactly, if $u_1 ... u_n \in \mathcal{F}$, then $t_u = u_n ... u_1$ and ${}^t\overline{u} = \overline{{}^t u} = (-1)^n u_n ... u_1$.

Definition 1.6. *The spinor norm of arbitrary* $u \in C(\mathcal{F})$ *is defined as* $S(u) = {}^t\overline{u} \cdot u \in C(\mathcal{F})$.

It is obvious that if $u, u', u'' \in \widetilde{\Gamma}(\mathcal{F})$, then $S(u, u') = S(u)S(u')$ and $S(uu'u'') = S(u)S(u')S(u'')$. For $u, u' \in \mathcal{F}S(u) = -G(u)$ and $S(u, u') = S(u)S(u') = S(uu')$.

Let us introduce the orthogonal group $O(G) \subset GL(G)$ defined by metric G on $\mathcal{F}$ and denote sets $SO(G) = \{u \in O(G), \det|u| = 1\}$, $Pin(G) = \{u \in \widetilde{\Gamma}(\mathcal{F}), S(u) = 1\}$ and $Spin(G) = Pin(G) \cap C^0(\mathcal{F})$. For $\mathcal{F} \cong \mathbb{R}^{n+m}$ we write $Spin(n+m)$. By straightforward calculations (see similar considerations in [Karoubi 1978]) we can verify the exactness of these sequences:

$$1 \to \mathbb{Z}/2 \to Pin(G) \to O(G) \to 1,$$

$$1 \to \mathbb{Z}/2 \to Spin(G) \to SO(G) \to 0,$$

$$1 \to \mathbb{Z}/2 \to Spin(n+m) \to SO(n+m) \to 1.$$

We conclude this section by emphasizing that the spinor norm was defined with respect to a quadratic form induced by a metric in v-bundle ξ_d (or by an H^{2n}-metric in the case of GL-spaces). This approach differs from those presented in [Asanov and Ponomarenko 1988] and [Ono and Takano 1993].

I.3 Clifford Bundles and Distinguished Spinor Structures

There are two possibilities for generalizing our spinor constructions defined for d-vector spaces to the case of vector bundle spaces enabled with the structure of N–connection. The firs is to use the extension to the category of vector bundles. The second is to define the Clifford fibration associated with compatible linear d-connection and metric G on a vector bundle (or with an H^{2n}-metric on GL-space). Let us consider both variants.

I.3.1 Clifford distinguished module structures in vector bundles.

Because functor $\mathcal{F} \to C(\mathcal{F})$ is smooth we can extend it to the category of vector bundles of type $\xi_d = \{\pi_d : HE \oplus VE \to E\}$. Recall that by $\mathcal{F}$ we denote

the typical fiber of such bundles. For ξ_d we obtain a bundle of algebras, denoted as $C(\xi_d)$, such that $C(\xi_d)_u = C(\mathcal{F}_u)$. Multiplication in every fibre defines a continuous map $C(\xi_d) \times C(\xi_d) \to C(\xi_d)$. If ξ_d is a vector bundle on number field k, the structure of the $C(\xi_d)$-module, the d-module, the d-module, on ξ_d is given by the continuous map $C(\xi_d) \times_E \xi_d \to \xi_d$ with every fiber $\mathcal{F}_u$ provided with the structure of the $C(\mathcal{F}_u)$–module, correlated with its k–module structure, Because $\mathcal{F} \subset C(\mathcal{F})$, we have a fiber to fiber map $\mathcal{F} \times_E \xi_d \to \xi_d$, inducing on every fiber the map $\mathcal{F}_u \times_E \xi_{d(u)} \to \xi_{d(u)}$ ($\mathbb{R}$–linear on the first factor and k-linear on the second one). Inversely, every such bilinear map defines on ξ_d the structure of the $C(\xi_d)$-module by virtue of universal properties of Clifford d–algebras. Equivalently, the above-mentioned bilinear map defines a morphism of v–bundles $m : \xi_d \to HOM(\xi_d, \xi_d)$ [$HOM(\xi_d, \xi_d)$ denotes the bundles of homomorphisms] when $(m(u))^2 = G(u)$ on every point.

Vector bundles ξ_d provided with $C(\xi_d)$–structures are objects of the category with morphisms being morphisms of v–bundles, which induce on every point $u \in \xi$ morphisms of $C(\mathcal{F}_u)$–modules. This is a Banach category contained in the category of finite-dimensional d-vector spaces on filed k. We shall not use category formalism in this work, but point to its advantages in further formulation of new directions of K–theory (see , for example, an introduction in [Karoubi 1978]) concerned with la–spaces.

Let us denote by $H^s(\xi, GL_{n+m}(\mathbb{R}))$ the s-dimensional cohomology group of the algebraic sheaf of germs of continuous maps of v–bundle ξ with group $GL_{n+m}(\mathbb{R})$ the group of automorphisms of $\mathbb{R}^{n+m}$ (for the language of algebraic topology see, for example, [Karoubi 1978] and [Godbillon 1971]). We shall also use the group $SL_{n+m}(\mathbb{R}) = \{A \subset GL_{n+m}(\mathbb{R}), \det A = 1\}$. Here we point out that cohomologies $H^s(M, Gr)$ characterize the class of a principal bundle $\pi : P \to M$ on M with structural group Gr. Taking into account that we deal with bundles distinguished by an N–connection we introduce into consideration cohomologies $H^s(\xi, GL_{n+m}(\mathbb{R}))$ as distinguished classes (d–classes) of bundles ξ provided with a global N–connection structure.

For a real vector bundle ξ_d on compact base ξ we can define the orientation on ξ_d as an element $\alpha_d \in H^1(\xi, GL_{n+m}(\mathbb{R}))$ whose image on map

$$H^1(\xi, SL_{n+m}(\mathbb{R})) \to H^1(\xi, GL_{n+m}(\mathbb{R}))$$

is the d–class of bundle ξ.

Definition 1.7. *The spinor structure on ξ_d is defined as an element $\beta_d \in H^1(\xi, Spin(n+m))$ whose image in the composition*

$$H^1(\xi, Spin(n+m)) \to H^1(\xi, SO(n+m)) \to H^1(\xi, GL_{n+m}(\mathbb{R}))$$

is the d–class of ξ.

The above definition of spinor structures can be reformulated in terms of principal bundles. Let ξ_d be a real vector bundle of rank n+m on a compact base ξ. If there is a principal bundle P_d with structural group $SO(n+m)$ [or $Spin(n+m)$],

this bundle ξ_d can be provided with orientation (or spinor) structure. The bundle P_d is associated with element $\alpha_d \in H^1(\xi, SO(n+m))$ [or $\beta_d \in H^1(\xi, Spin(n+m))$.

We remark that a real bundle is oriented if and only if its first Stiefel–Whitney d–class vanishes,

$$w_1(\xi_d) \in H^1(\xi, \mathbb{Z}/2) = 0,$$

where $H^1(\xi, \mathbb{Z}/2)$ is the first group of Chech cohomology with coefficients in $\mathbb{Z}/2$. Considering the second Stiefel-Whitney class $w_2(\xi_d) \in H^{21}(\xi, \mathbb{Z}/2)$ it is well known that vector bundle ξ_d admits the spinor structure if and only if $w_2(\xi_d) = 0$. Finally, in this subsection, we emphasize that taking into account that base space ξ is also a v-bundle, $p : E \to M$, we have to make explicit calculations in order to express cohomologies $H^s(\xi, GL_{n+m})$ and $H^s(\xi, SO(n+m))$ through cohomologies $H^s(M, GL_n), H^s(M, SO(m))$, which depends on global topological structures of spaces M and ξ. For general bundle and base spaces this requires a cumbersome cohomological calculus.

I.3.2 Clifford fibration.

Another way of defining the spinor structure is to use Clifford fibrations. Consider the principal bundle with the structural group Gr being a subgroup of orthogonal group $O(G)$, where G is a quadratic nondegenerate form (see(1.12)) defined on the base (also being a bundle space) space ξ. The fibration associated to principal fibration $P(\xi, Gr)$ $\left[\text{or } P\left(H^{2n}, Gr\right)\right]$ with a typical fiber having Clifford algebra $C(G)$ is, by definition, the Clifford fibration $PC(\xi, Gr)$. We can always define a metric on the Clifford fibration if every fiber is isometric to $PC(\xi, G)$ (this result is proved for arbitrary quadratic forms G on pseudo-Riemannian bases [Turtoi 1989]). If, additionally, $Gr \subset SO(G)$ a global section can be defined on $PC(G)$.

Let $\mathcal{P}(\xi, Gr)$ be the set of principal bundles with differentiable base ξ and structural group Gr. If $g : Gr \to Gr'$ is an homomorphism of Lie groups and $P(\xi, Gr) \subset \mathcal{P}(\xi, Gr)$ (for simplicity in this section we shall denote mentioned bundles and sets of bundles as P, P' and respectively, $\mathcal{P}, \mathcal{P}'$), we can always construct a principal bundle with the property that there is as homomorphism $f : P' \to P$ of principal bundles which can be projected to the identity map of ξ and corresponds to isomorphism $g : Gr \to Gr'$. If the inverse statement also holds, the bundle P' is called as the extension of P associated to g and f is called the extension homomorphism denoted as $\widetilde{g}$.

Now we can define distinguished spinor structures on bundle spaces (compare with definition 1.7).

Definition 1.8. *Let $P \in \mathcal{P}(\xi, O(G))$ be a principal bundle. A distinguished spinor structure of P, equivalently a ds-structure of ξ is an extension $\widetilde{P}$ of P associated to homomorphism $h : PinG \to O(G)$ where $O(G)$ is the group of orthogonal rotations, generated by metric G, in bundle ξ.*

So, if $\widetilde{P}$ is a spinor structure of the space ξ, then $\widetilde{P} \in \mathcal{P}(\xi, PinG)$.

On the definition of spinor structures on varieties we cite [Geroch 1958]. It is proved that a necessary and sufficient condition for a space time to be orientable is to admit a global field of orthonormalized frames. We mention that spinor structures

can be also defined on varieties modeled on Banach spaces [Anastasiei 1977]. As we have shown in this subsection, similar constructions are possible for the cases when space time has the structure of a v–bundle with an N–connection.

Definition 1.9. *A special distinguished spinor structure, ds-structure, of principal bundle* $P = P(\xi, SO(G))$ *is a principal bundle* $\widetilde{P} = \widetilde{P}(\xi, SpinG)$ *for which a homomorphism of principal bundles* $\widetilde{p} : \widetilde{P} \to P$*, projected on the identity map of* ξ *(or of* H^{2n}*) and corresponding to representation*

$$R : SpinG \to SO(G),$$

is defined.

In the case when the base space variety is oriented, there is a natural bijection between tangent spinor structures with a common base. For special ds–structures we can define, as for any spinor structure, the concepts of spin tensors, spinor connections, and spinor covariant derivations.

I.4 Almost Complex Spinor Structures

Almost complex structures are an important characteristic of H^{2n}-spaces. We can rewrite the almost Hermitian metric H^{2n}-metric (see considerations from subsection 1.1.1 with respect to conditions of type (1.10) and (1.16) for a metric (1.12)), in complex form:

$$G = H_{ab}(z, \xi)\, dz^a \otimes dz^b, \tag{1.47}$$

where

$$z^a = x^a + iy^a,\ \overline{z^a} = x^a - iy^a,\ H_{ab}(z, \overline{z}) = g_{ab}(x, y)\mid^{x=x(z,\overline{z})}_{y=y(z,\overline{z})},$$

and define almost complex spinor structures. For given metric (1.47) on H^{2n}-space there is always a principal bundle P^U with unitary structural group U(n) which allows us to transform H^{2n}-space into v-bundle $\xi^U \approx P^U \times_{U(n)} \mathbb{R}^{2n}$. This statement will be proved after we introduce complex spinor structures on oriented real vector bundles [Karoubi 1978].

Let us consider momentarily $k = \mathcal{C}$ and introduce into consideration [instead of the group $Spin(n)$] the group $Spin^c \times_{\mathbb{Z}/2} U(1)$ being the factor group of the product $Spin(n) \times U(1)$ with the respect to equivalence

$$(y, z) \sim (-y, -a), \quad y \in Spin(m).$$

This way we define the short exact sequence

$$1 \to U(1) \to Spin^c(n) \xrightarrow{\rho^c} SO(n) \to 1,$$

where $\rho^c(y, a) = \rho^c(y)$. If λ is oriented , real, and rank n, γ-bundle $\pi : E_\lambda \to M^n$, with base M^n, the complex spinor structure, spin structure, on λ is given

by the principal bundle P with structural group $Spin^c(m)$ and isomorphism $\lambda \approx P \times_{Spin^c(n)} \mathbb{R}^n$. For such bundles the categorial equivalence can be defined as

$$\epsilon^c : \mathcal{E}^T_{\mathbb{C}}(M^n) \to \mathcal{E}^\lambda_{\mathbb{C}}(M^n), \tag{1.48}$$

where $\epsilon^c(E^c) = P \triangle_{Spin^c(n)} E^c$ is the category of trivial complex bundles on $M^n, \mathcal{E}^\lambda_{\mathbb{C}}(M^n)$ is the category of complex v-bundles on M^n with action of Clifford bundle $C(\lambda), P\triangle_{Spin^c(n)}$ and E^c is the factor space of the bundle product $P \times_M E^c$ with respect to the equivalence $(p,e) \sim \left(p\widehat{g}^{-1}, \widehat{g}e\right), p \in P, e \in E^c$, where $\widehat{g} \in Spin^c(n)$ acts on E by via the imbedding $Spin(n) \subset C^{0,n}$ and the natural action $U(1) \subset \mathcal{C}$ on complex v-bundle $\xi^c, E^c = tot\xi^c$, for bundle $\pi^c : E^c \to M^n$.

Now we return to the bundle ξ. A real v-bundle (not being a spinor bundle) admits a complex spinor structure if and only if there exist a homomorphism $\sigma : U(n) \to Spin^c(2n)$ making the diagram

$$\begin{array}{ccc} U(n) & \xrightarrow{\sigma} & Spin^c(2n) \\ \varphi\big\downarrow & & \big\downarrow\rho^c \\ SO(2n) & \Longleftrightarrow & SO(2n) \end{array} \tag{1.49}$$

commutative. The explicit construction of σ for arbitrary γ–bundle is given in [Karoubi 1978] and [Atiyah, Bott and Shapiro 1964]. For H^{2n}-spaces it is obvious that a diagram similar to (1.49) can be defined for the tangent bundle TM^n, which directly points to the possibility of defining the ${}^cSpin$-structure on H^{2n}-spaces.

Let λ be a complex, rank n, spinor bundle with

$$\tau : Spin^c(n) \times_{\mathbb{Z}/2} U(1) \to U(1) \tag{1.50}$$

the homomorphism defined by formula $\tau(\lambda, \delta) = \delta^2$. For P_s being the principal bundle with fiber $Spin^c(n)$ we introduce the complex linear bundle $L(\lambda^c) = P_S \times_{Spin^c(n)} \mathbb{C}$ defined as the factor space of $P_S \times \mathbb{C}$ on equivalence relation

$$(pt, z) \sim \left(p, l(t)^{-1} z\right),$$

where $t \in Spin^c(n)$. This linear bundle is associated to complex spinor structure on λ^c.

If λ^c and $\lambda^{c'}$ are complex spinor bundles, the Whitney sum $\lambda^c \oplus \lambda^{c'}$ is naturally provided with the structure of the complex spinor bundle. This follows from the holomorphism

$$\omega' : Spin^c(n) \times Spin^c(n') \to Spin^c(n+n'), \tag{1.51}$$

given by formula $[(\beta, z), (\beta', z')] \to [\omega(\beta, \beta'), zz']$, where ω is the homomorphism making the following diagram commutative:

$$\begin{array}{ccc} Spin(n) \times Spin(n') & \longrightarrow & Spin(n+n') \\ \big\downarrow & & \big\downarrow \\ O(n) \times O(n') & \longrightarrow & O(n+n') \end{array}$$

Here, $z, z' \in U(1)$. It is obvious that $L\left(\lambda^c \oplus \lambda^{c'}\right)$ is isomorphic to $L(\lambda^c) \otimes L\left(\lambda^{c'}\right)$.

We conclude this section by formulating our main result on complex spinor structures for H^{2n}-spaces:

Theorem 1.3. *Let λ^c be a complex spinor bundle of rank n and H^{2n}-space considered as a real vector bundle $\lambda^c \oplus \lambda^{c'}$ provided with almost complex structure $J^\alpha{}_\beta$; multiplication on i is given by $\begin{pmatrix} 0 & -\delta^i_j \\ \delta^i_j & 0 \end{pmatrix}$. Then, the diagram*

$$\begin{array}{ccc} \mathcal{E}^{0,2n}_{\mathbb{C}}(M^{2n}) & \xrightarrow{\varepsilon^c} & \mathcal{E}^{\lambda^c \oplus \lambda^c}(M^n) \\ {\scriptstyle \tilde{\varepsilon}^c}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{H}} \\ \mathcal{E}^W_{\mathbb{C}}(M^n) & \Longleftrightarrow & \mathcal{E}^W_{\mathbb{C}}(M^n) \end{array}$$

is commutative up to isomorphisms ϵ^c and $\widetilde{\epsilon}^c$ defined as in (1.48), $\mathcal{H}$ is functor $E^c \to E^c \otimes L(\lambda^c)$ and $\mathcal{E}^{0,2n}_{\mathbb{C}}(M^n)$ is defined by functor $\mathcal{E}_{\mathbb{C}}(M^n) \to \mathcal{E}^{0,2n}_{\mathbb{C}}(M^n)$ given as correspondence $E^c \to \Lambda(\mathbb{C}^n) \otimes E^c$ (which is a categorial equivalence), $\Lambda(\mathbb{C}^n)$ is the exterior algebra on $\mathbb{C}^n$. W is the real bundle $\lambda^c \oplus \lambda^{c'}$ provided with complex structure.

Proof: We use composition of homomorphisms

$$\mu : Spin^c(2n) \xrightarrow{\pi} SO(n) \xrightarrow{r} U(n) \xrightarrow{\sigma} Spin^c(2n) \times_{\mathbb{Z}/2} U(1),$$

commutative diagram

$$\begin{array}{ccc} Spin(2n) & \subset & Spin^c(2n) \\ {\scriptstyle \beta}\big\uparrow & & \big\uparrow \\ SO(n) & \longrightarrow & SO(2n) \end{array}$$

and introduce composition of homomorphisms

$$\mu : Spin^c(n) \xrightarrow{\Delta} Spin^c(n) \times Spin^c(n) \xrightarrow{\omega^c} Spin^c(n),$$

where Δ is the diagonal homomorphism and ω^c is defined as in (1.51). Using homomorphisms (1.50) and (1.51) we obtain formula $\mu(t) = \mu(t)\, r(t)$.

Now consider bundle $P \times_{Spin^c(n)} Spin^c(2n)$ as the principal $Spin^c(2n)$-bundle, associated to $M \oplus M$ being the factor space of the product $P \times Spin^c(2n)$ on the equivalence relation $(p, t, h) \sim \left(p, \mu(t)^{-1} h\right)$. In this case the categorial equivalence (1.49) can be rewritten as

$$\epsilon^c(E^c) = P \times_{Spin^c(n)} Spin^c(2n)\, \Delta_{Spin^c(2n)} E^c$$

and seen as factor space of $P \times Spin^c(2n) \times_M E^c$ on equivalence relation

$$(pt, h, e) \sim \left(p, \mu(t)^{-1} h, e\right) \text{ and } (p, h_1, h_2, e) \sim \left(p, h_1, h_2^{-1} e\right)$$

(projections of elements p and e coincides on base M). Every element of $\epsilon^c(E^c)$ can be represented as $P \Delta_{Spin^c(n)} E^c$, i.e., as a factor space $P \Delta E^c$ on equivalence

relation $(pt, e) \sim (p, \mu^c(t), e)$, when $t \in Spin^c(n)$. The complex line bundle $L(\lambda^c)$ can be interpreted as the factor space of $P \times_{Spin^c(n)} \mathbb{C}$ on equivalence relation $(pt, \delta) \sim \left(p, r(t)^{-1} \delta\right)$.

Putting $(p, e) \otimes (p, \delta)(p, \delta e)$ we introduce morphism

$$\epsilon^c(E) \times L(\lambda^c) \rightarrow \epsilon^c(\lambda^c)$$

with properties $(pt, e) \otimes (pt, \delta) \rightarrow (pt, \delta e) = \left(p, \mu^c(t)^{-1} \delta e\right)$,

$\left(p, \mu^c(t)^{-1} e\right) \otimes \left(p, l(t)^{-1} e\right) \rightarrow \left(p, \mu^c(t) r(t)^{-1} \delta e\right)$ pointing to the fact that we have defined the isomorphism correctly and that it is an isomorphism on every fiber. ■

I.5 Spinor Techniques for Distinguished Vector Spaces

The purpose of this section is to show how a corresponding abstract spinor technique entailing notational and calculations advantages can be developed for arbitrary splits of dimensions of a d-vector space $\mathcal{F} = h\mathcal{F} \oplus v\mathcal{F}$, where $\dim h\mathcal{F} = n$ and $\dim v\mathcal{F} = m$. For convenience we shall also present some necessary coordinate expressions.

The problem of a rigorous definition of spinors on la-spaces (la-spinors, d-spinors) was posed and solved [Vacaru 1996] (see previous sections 1.2–1.4) in the framework of the formalism of Clifford and spinor structures on v-bundles provided with compatible nonlinear and distinguished connections and metric. We introduced d–spinors as corresponding objects of the Clifford d–algebra $\mathcal{C}(\mathcal{F}, G)$, defined for a d-vector space $\mathcal{F}$ in a standard manner (see, for instance, [Karoubi 1978]) and proved that operations with $\mathcal{C}(\mathcal{F}, G)$ can be reduced to calculations for $\mathcal{C}(h\mathcal{F}, g)$ and $\mathcal{C}(v\mathcal{F}, h)$, which are usual Clifford algebras of respective dimensions 2^n and 2^m (if it is necessary we can use quadratic forms g and h correspondingly induced on $h\mathcal{F}$ and $v\mathcal{F}$ by a metric $\mathbf{G}$ (1.12)). Considering the orthogonal subgroup $O(\mathbf{G}) \subset GL(\mathbf{G})$ defined by a metric $\mathbf{G}$ we can define the d-spinor norm and parametrize d-spinors by ordered pairs of elements of Clifford algebras $\mathcal{C}(h\mathcal{F}, g)$ and $\mathcal{C}(v\mathcal{F}, h)$. We emphasize that the splitting of a Clifford d-algebra associated to a v-bundle $\mathcal{E}$ is a straightforward consequence of the global decomposition (1.3) defining a N-connection structure in $\mathcal{E}$.

In this section, as a rule, we shall omit proofs which in most cases are mechanical but rather tedious. We can apply the methods developed in [Penrose and Rindler 1984, 1986] and [Luehr and Rosenbaum 1974] in a straightforward manner on h- and v-subbundles in order to verify the correctness of affirmations.

I.5.1 Clifford d-algebra, d-spinors and d-twistors.

In order to relate the succeeding constructions with Clifford d-algebras we consider a la–frame decomposition of the metric (1.12):

$$G_{\alpha\beta}(u) = l_{\alpha}^{\widehat{\alpha}}(u)\, l_{\beta}^{\widehat{\beta}}(u)\, G_{\widehat{\alpha}\widehat{\beta}}, \tag{1.52}$$

where the frame d-vectors and constant metric matrices are respectively distinguished as

$$l_{\alpha}^{\widehat{\alpha}}(u)=\begin{pmatrix} l_{j}^{\widehat{j}}(u) & 0 \\ 0 & l_{a}^{\widehat{a}}(u) \end{pmatrix} \text{ and } G_{\widehat{\alpha}\widehat{\beta}}\begin{pmatrix} g_{\widehat{i}\widehat{j}} & 0 \\ 0 & h_{\widehat{a}\widehat{b}} \end{pmatrix}, \tag{1.53}$$

where $g_{\widehat{i}\widehat{j}}$ and $h_{\widehat{a}\widehat{b}}$ are diagonal matrices with $g_{\widehat{i}\widehat{i}}=h_{\widehat{a}\widehat{a}}=\pm 1$.

To generate Clifford d-algebras we start with matrix equations

$$\sigma_{\widehat{\alpha}}\sigma_{\widehat{\beta}}+\sigma_{\widehat{\beta}}\sigma_{\widehat{\alpha}}=-G_{\widehat{\alpha}\widehat{\beta}}I, \tag{1.54}$$

where I is the identity matrix, matrices $\sigma_{\widehat{\alpha}}$ (σ-objects) act on a d-vector space $\mathcal{F}=h\mathcal{F}\oplus v\mathcal{F}$ and theirs components are distinguished as

$$\sigma_{\widehat{\alpha}}=\left\{(\sigma_{\widehat{\alpha}})_{\underline{\beta}}^{\cdot\underline{\gamma}}=\begin{pmatrix} (\sigma_{\widehat{i}})_{\underline{j}}^{\cdot\underline{k}} & 0 \\ 0 & (\sigma_{\widehat{a}})_{\underline{b}}^{\cdot\underline{c}} \end{pmatrix}\right\}, \tag{1.55}$$

indices $\underline{\beta},\underline{\gamma},...$ refer to spin spaces of type $\mathcal{S}=S_{(h)}\oplus S_{(v)}$ and underlined Latin indices $\underline{j},\underline{k},...$ and $\underline{b},\underline{c},...$ refer respectively to a h-spin space $\mathcal{S}_{(h)}$ and a v-spin space $\mathcal{S}_{(v)}$, which are correspondingly associated to a h- and v-decomposition of a v-bundle $\mathcal{E}_{(d)}$. The irreducible algebra of matrices $\sigma_{\widehat{\alpha}}$ of minimal dimension $N\times N$, where $N=N_{(n)}+N_{(m)}$, $\dim\mathcal{S}_{(h)}=N_{(n)}$ and $\dim\mathcal{S}_{(v)}=N_{(m)}$, has these dimensions

$$N_{(n)}=\begin{cases} 2^{(n-1)/2}, & \text{for } n=2k+1 \\ 2^{n/2}, & \text{for } n=2k \end{cases}$$

and

$$N_{(m)}=\begin{cases} 2^{(m-1)/2}, & \text{for } m=2k+1 \\ 2^{m/2}, & \text{for } m=2k, \end{cases}$$

where $k=1,2,...$.

The Clifford d-algebra is generated by sums on $n+1$ elements of form

$$A_1 I+B^{\widehat{i}}\sigma_{\widehat{i}}+C^{\widehat{i}\widehat{j}}\sigma_{\widehat{i}\widehat{j}}+D^{\widehat{i}\widehat{j}\widehat{k}}\sigma_{\widehat{i}\widehat{j}\widehat{k}}+...$$

and sums of $m+1$ elements of form

$$A_2 I+B^{\widehat{a}}\sigma_{\widehat{a}}+C^{\widehat{a}\widehat{b}}\sigma_{\widehat{a}\widehat{b}}+D^{\widehat{a}\widehat{b}\widehat{c}}\sigma_{\widehat{a}\widehat{b}\widehat{c}}+...$$

with antisymmetric coefficients $C^{\widehat{i}\widehat{j}}=C^{[\widehat{i}\widehat{j}]}, C^{\widehat{a}\widehat{b}}=C^{[\widehat{a}\widehat{b}]}, D^{\widehat{i}\widehat{j}\widehat{k}}=D^{[\widehat{i}\widehat{j}\widehat{k}]}, D^{\widehat{a}\widehat{b}\widehat{c}}=D^{[\widehat{a}\widehat{b}\widehat{c}]},...$ and matrices $\sigma_{\widehat{i}\widehat{j}}=\sigma_{[\widehat{i}}\sigma_{\widehat{j}]}, \sigma_{\widehat{a}\widehat{b}}=\sigma_{[\widehat{a}}\sigma_{\widehat{b}]}, \sigma_{\widehat{i}\widehat{j}\widehat{k}}=\sigma_{[\widehat{i}}\sigma_{\widehat{j}}\sigma_{\widehat{k}]},...$. Really, we have 2^{n+1} coefficients $\left(A_1, C^{\widehat{i}\widehat{j}}, D^{\widehat{i}\widehat{j}\widehat{k}},...\right)$ and 2^{m+1} coefficients $\left(A_2, C^{\widehat{a}\widehat{b}}, D^{\widehat{a}\widehat{b}\widehat{c}},...\right)$ of the Clifford algebra on $\mathcal{F}$.

For simplicity, in this subsection, we shall present the necessary geometric constructions only for h-spin spaces $\mathcal{S}_{(h)}$ of dimension $N_{(n)}$. Considerations for a v-spin space $\mathcal{S}_{(v)}$ are similar but with proper characteristics for a dimension $N_{(m)}$.

In order to define the scalar (spinor) product on $\mathcal{S}_{(h)}$ we introduce into consideration this finite sum (because of a finite number of elements $\sigma_{[\widehat{ij...\widehat{k}}]}$):

$$(1.56)\quad {}^{(\pm)}E^{\underline{ij}}_{\underline{km}} = \delta^{\underline{i}}_{\underline{k}}\delta^{\underline{j}}_{\underline{m}} + \frac{2}{1!}(\sigma_{\widehat{i}})^{\cdot \underline{i}}_{\underline{k}}(\sigma^{\widehat{i}})^{\cdot \underline{j}}_{\underline{m}} + \frac{2^2}{2!}(\sigma_{\widehat{\widehat{ij}}})^{\cdot \underline{i}}_{\underline{k}}(\sigma^{\widehat{\widehat{ij}}})^{\cdot \underline{j}}_{\underline{m}} + \frac{2^3}{3!}(\sigma_{\widehat{\widehat{\widehat{ijk}}}})^{\cdot \underline{i}}_{\underline{k}}(\sigma^{\widehat{\widehat{\widehat{ijk}}}})^{\cdot \underline{j}}_{\underline{m}} + ...$$

which can be factorized as

$$(1.57)\qquad {}^{(\pm)}E^{\underline{ij}}_{\underline{km}} = N_{(n)}\,{}^{(\pm)}\epsilon_{\underline{km}}\,{}^{(\pm)}\epsilon^{\underline{ij}} \text{ for } n = 2k$$

and

$$(1.58)\qquad {}^{(+)}E^{\underline{ij}}_{\underline{km}} = 2N_{(n)}\epsilon_{\underline{km}}\epsilon^{\underline{ij}},\ {}^{(-)}E^{\underline{ij}}_{\underline{km}} = 0 \text{ for } n = 3(mod4),$$

$$ {}^{(+)}E^{\underline{ij}}_{\underline{km}} = 0,\ {}^{(-)}E^{\underline{ij}}_{\underline{km}} = 2N_{(n)}\epsilon_{\underline{km}}\epsilon^{\underline{ij}} \text{ for } n = 1(mod4).$$

Antisymmetry of $\sigma_{\widehat{\widehat{\widehat{ijk...}}}}$ and the construction of the objects (1.56),(1.57) and (1.58) define the properties of ϵ-objects ${}^{(\pm)}\epsilon_{\underline{km}}$ and $\epsilon_{\underline{km}}$ which have an eight–fold periodicity on n (see details in [Penrose and Rindler 1986] and, with respect to la–spaces, [Vacaru 1986]).

For even values of n it is possible the decomposition of every h-spin space $\mathcal{S}_{(h)}$into irreducible h-spin spaces $\mathbf{S}_{(h)}$ and $\mathbf{S}'_{(h)}$ (one considers splitting of h-indices, for instance, $\underline{l} = L \oplus L', \underline{m} = M \oplus M', ...$; for v-indices we shall write $\underline{a} = A \oplus A', \underline{b} = B \oplus B', ...$) and defines new ϵ-objects

$$(1.59)\qquad \epsilon^{\underline{lm}} = \frac{1}{2}\left({}^{(+)}\epsilon^{\underline{lm}} + {}^{(-)}\epsilon^{\underline{lm}}\right) \text{ and } \widetilde{\epsilon}^{\underline{lm}} = \frac{1}{2}\left({}^{(+)}\epsilon^{\underline{lm}} - {}^{(-)}\epsilon^{\underline{lm}}\right)$$

We shall omit similar formulas for ϵ-objects with lower indices.

We can verify, by using expressions (1.58) and straightforward calculations, these parametrizations on symmetry properties of ϵ-objects (1.59)

$$(1.60)\quad \epsilon^{\underline{lm}} = \begin{pmatrix} \epsilon^{LM} = \epsilon^{ML} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \widetilde{\epsilon}^{\underline{lm}} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{\epsilon}^{LM} = \widetilde{\epsilon}^{ML} \end{pmatrix}$$

for $n = 0(mod8)$;

$$\epsilon^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = \epsilon^{\underline{ml}}, \text{ where } {}^{(+)}\epsilon^{\underline{lm}} = 0 \text{ and } \widetilde{\epsilon}^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = \widetilde{\epsilon}^{\underline{ml}},$$

for $n = 1(mod8)$;

$$\epsilon^{\underline{lm}} = \begin{pmatrix} 0 & 0 \\ \epsilon^{L'M} & 0 \end{pmatrix} \text{ and } \widetilde{\epsilon}^{\underline{lm}} = \begin{pmatrix} 0 & \widetilde{\epsilon}^{LM'} = -\epsilon^{M'L} \\ 0 & 0 \end{pmatrix}$$

for $n = 2(mod8)$;

$$\epsilon^{\underline{lm}} = -\frac{1}{2}{}^{(+)}\epsilon^{\underline{lm}} = -\epsilon^{\underline{ml}}, \text{ where } {}^{(-)}\epsilon^{\underline{lm}} = 0 \text{ and } \widetilde{\epsilon}^{\underline{lm}} = \frac{1}{2}{}^{(+)}\epsilon^{\underline{lm}} = -\widetilde{\epsilon}^{\underline{ml}},$$
$$\text{for } n = 3(mod8);$$

$$\epsilon^{\underline{lm}} = \begin{pmatrix} \epsilon^{LM} = -\epsilon^{ML} & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \widetilde{\epsilon}^{\underline{lm}} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{\epsilon}^{LM} = -\widetilde{\epsilon}^{ML} \end{pmatrix}$$
$$\text{for } n = 4(mod8);$$

$$\epsilon^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = -\epsilon^{\underline{ml}}, \text{ where } {}^{(+)}\epsilon^{\underline{lm}} = 0 \text{ and } \widetilde{\epsilon}^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = -\widetilde{\epsilon}^{\underline{ml}},$$
$$\text{for } n = 5(mod8);$$

$$\epsilon^{\underline{lm}} = \begin{pmatrix} 0 & 0 \\ \epsilon^{L'M} & 0 \end{pmatrix} \text{ and } \widetilde{\epsilon}^{\underline{lm}} = \begin{pmatrix} 0 & \widetilde{\epsilon}^{LM'} = \epsilon^{M'L} \\ 0 & 0 \end{pmatrix}$$
$$\text{for } n = 6(mod8);$$

$$\epsilon^{\underline{lm}} = \frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = \epsilon^{\underline{ml}}, \text{ where } {}^{(+)}\epsilon^{\underline{lm}} = 0 \text{ and } \widetilde{\epsilon}^{\underline{lm}} = -\frac{1}{2}{}^{(-)}\epsilon^{\underline{lm}} = \widetilde{\epsilon}^{\underline{ml}},$$
$$\text{for } n = 7(mod8).$$

Let denote reduced and irreducible h-spinor spaces in a form pointing to the symmetry of spinor inner products in dependence of values $n = 8k+l$ $(k = 0, 1, 2, ...; l = 1, 2, ...7)$ of the dimension of the horizontal subbundle (we shall write respectively $\triangle$ and $\circ$ for antisymmetric and symmetric inner products of reduced spinors and $\diamondsuit = (\triangle, \circ)$ and $\widetilde{\diamondsuit} = (\circ, \triangle)$ for corresponding parametrizations of inner products, in brief *i.p.*, of irreducible spinors; properties of scalar products of spinors are defined by ϵ-objects (1.60); we shall use $\blacklozenge$ for a general *i.p.* when the symmetry is not pointed out):

$$\mathcal{S}_{(h)}\,(8k) = \mathbf{S}_\circ \oplus \mathbf{S}'_\circ; \tag{1.61}$$

$$\mathcal{S}_{(h)}\,(8k+1) = \mathcal{S}^{(-)}_\circ \text{ (}\textit{i.p.}\text{ is defined by an } {}^{(-)}\epsilon\text{-object);}$$

$$\mathcal{S}_{(h)}\,(8k+2) = \begin{cases} \mathcal{S}_\blacklozenge = (\mathbf{S}_\blacklozenge, \mathbf{S}_\blacklozenge), & \text{or} \\ \mathcal{S}'_\blacklozenge = (\mathbf{S}'_{\widetilde{\blacklozenge}}, \mathbf{S}'_{\widetilde{\blacklozenge}}) & ; \end{cases}$$

$$\mathcal{S}_{(h)}\,(8k+3) = \mathcal{S}^{(+)}_\triangle \text{ (}\textit{i.p.}\text{ is defined by an } {}^{(+)}\epsilon\text{-object);}$$

$$\mathcal{S}_{(h)}\,(8k+4) = \mathbf{S}_\triangle \oplus \mathbf{S}'_\triangle;$$

$$\mathcal{S}_{(h)}\,(8k+5) = \mathcal{S}^{(-)}_\triangle \text{ (}\textit{i.p.}\text{ is defined by an } {}^{(-)}\epsilon\text{-object),}$$

$$\mathcal{S}_{(h)}\,(8k+6) = \begin{cases} \mathcal{S}_\blacklozenge = (\mathbf{S}_\blacklozenge, \mathbf{S}_\blacklozenge), & \text{or} \\ \mathcal{S}'_\blacklozenge = (\mathbf{S}'_{\widetilde{\blacklozenge}}, \mathbf{S}'_{\widetilde{\blacklozenge}}) & ; \end{cases}$$

$$\mathcal{S}_{(h)}\,(8k+7) = \mathcal{S}_{\circ}^{(+)} \ (i.p. \quad \text{is defined by an } {}^{(+)}\epsilon\text{-object}).$$

We note that by using corresponding ϵ-objects we can lower and rise indices of reduced and irreducible spinors (for $n = 2, 6(mod 4)$ we can exclude primed indices, or inversely, see details in [Penrose and Rindler 1986]).

The similar v-spinor spaces are denoted by the same symbols as in (1.61) provided with a left lower mark "$|$" and parametrized with respect to the values $m = 8k' + l$ (k'=0,1,...; l=1,2,...,7) of the dimension of the vertical subbundle, for example, as

$$\mathcal{S}_{(v)}(8k') = \mathbf{S}_{|\circ} \oplus \mathbf{S}'_{|\circ}, \mathcal{S}_{(v)}\,(8k+1) = \mathcal{S}_{|\circ}^{(-)}, ... \tag{1.62}$$

We use "$\widetilde{\ }$"-overlined symbols,

$$\widetilde{\mathcal{S}}_{(h)}\,(8k) = \widetilde{\mathbf{S}}_{\circ} \oplus \widetilde{S}'_{\circ}, \widetilde{\mathcal{S}}_{(h)}\,(8k+1) = \widetilde{\mathcal{S}}_{\circ}^{(-)}, ... \tag{1.63}$$

and

$$\widetilde{\mathcal{S}}_{(v)}(8k') = \widetilde{\mathbf{S}}_{|\circ} \oplus \widetilde{S}'_{|\circ}, \widetilde{\mathcal{S}}_{(v)}\,(8k'+1) = \widetilde{\mathcal{S}}_{|\circ}^{(-)}, ... \tag{1.64}$$

respectively for the dual to (1.61) and (1.62) spinor spaces.

The spinor spaces (1.61)-(1.64) are called the prime spinor spaces, in brief p-spinors. They are considered as building blocks of distinguished (n,m)-spinor spaces constructed in this manner:

$$\mathcal{S}(_{\circ\circ,\circ\circ}) = \mathbf{S}_{\circ} \oplus \mathbf{S}'_{\circ} \oplus \mathbf{S}_{|\circ} \oplus \mathbf{S}'_{|\circ}, \mathcal{S}(_{\circ\circ,\circ}|^{\circ}) = \mathbf{S}_{\circ} \oplus \mathbf{S}'_{\circ} \oplus \mathbf{S}_{|\circ} \oplus \widetilde{S}'_{|\circ}, \tag{1.65}$$

$$\mathcal{S}(_{\circ\circ,}|^{\circ\circ}) = \mathbf{S}_{\circ} \oplus \mathbf{S}'_{\circ} \oplus \widetilde{S}_{|\circ} \oplus \widetilde{S}'_{|\circ}, \mathcal{S}(_{\circ}|^{\circ\circ\circ}) = \mathbf{S}_{\circ} \oplus \widetilde{S}'_{\circ} \oplus \widetilde{S}_{|\circ} \oplus \widetilde{S}'_{|\circ},$$

....

$$\mathcal{S}(_{\triangle,\triangle}) = \mathcal{S}_{\triangle}^{(+)} \oplus S_{|\triangle}^{(+)}, S(_{\triangle,}{}^{\triangle}) = \mathcal{S}_{\triangle}^{(+)} \oplus \widetilde{S}_{|\triangle}^{(+)},$$

....

$$\mathcal{S}(_{\triangle}|^{\circ},{}_{\diamondsuit}) = \mathbf{S}_{\triangle} \oplus \widetilde{S_{\circ}}' \oplus \mathcal{S}_{|\diamondsuit}, \mathcal{S}(_{\triangle}|^{\circ},{}^{\diamondsuit}) = \mathbf{S}_{\triangle} \oplus \widetilde{\mathcal{S}_{\circ}}' \oplus \widetilde{\mathcal{S}}_{|}^{\diamondsuit},$$

....

Considering the operation of dualization of prime components in (1.65) we can generate different isomorphic variants of distinguished (n,m)-spinor spaces.

We define a d-spinor space $\mathcal{S}_{(n,m)}$ as a direct sum of a horizontal and a vertical spinor spaces of type (1.64), for instance,

$$\mathcal{S}_{(8k,8k')} = \mathbf{S}_{\circ} \oplus \mathbf{S}'_{\circ} \oplus \mathbf{S}_{|\circ} \oplus \mathbf{S}'_{|\circ}, \mathcal{S}_{(8k,8k'+1)} = \mathbf{S}_{\circ} \oplus \mathbf{S}'_{\circ} \oplus \mathcal{S}_{|\circ}^{(-)}, ...,$$

$$\mathcal{S}_{(8k+4,8k'+5)} = \mathbf{S}_{\triangle} \oplus \mathbf{S}'_{\triangle} \oplus \mathcal{S}_{|\triangle}^{(-)}, ...$$

The scalar product on a $\mathcal{S}_{(n,m)}$ is induced by (corresponding to fixed values of n and m) ϵ-objects (1.60) considered for h- and v-components.

Having introduced d-spinors for dimensions (n,m) we can write out the generalization for la-spaces of twistor equations [Penrose and Rindler 1986] by using the distinguished σ-objects (1.55):

$$(\sigma_{(\widehat{\alpha}})^{\cdot\cdot\underline{\gamma}}_{|\underline{\beta}|}\quad \frac{\delta\omega^{\underline{\beta}}}{\delta u^{\widehat{\beta})}} = \frac{1}{n+m}\quad G_{\widehat{\alpha}\widehat{\beta}}(\sigma^{\widehat{\epsilon}})^{\cdot\cdot\underline{\gamma}}_{\underline{\beta}}\quad \frac{\delta\omega^{\underline{\beta}}}{\delta u^{\widehat{\epsilon}}}, \tag{1.66}$$

where $\left|\underline{\beta}\right|$ denotes that we do not consider symmetrization on this index. The general solution of (1.66) on the d-vector space $\mathcal{F}$ looks like as

$$\omega^{\underline{\beta}} = \Omega^{\underline{\beta}} + u^{\widehat{\alpha}}(\sigma_{\widehat{\alpha}})^{\cdot\cdot\underline{\beta}}_{\underline{\epsilon}}\Pi^{\underline{\epsilon}}, \tag{1.67}$$

where $\Omega^{\underline{\beta}}$ and $\Pi^{\underline{\epsilon}}$ are constant d-spinors. For fixed values of dimensions n and m we mast analyze the reduced and irreducible components of h- and v-parts of equations (1.66) and their solutions (1.67) in order to find the symmetry properties of a d-twistor $\mathbf{Z}^{\alpha}$ defined as a pair of d-spinors

$$\mathbf{Z}^{\alpha} = (\omega^{\underline{\alpha}}, \pi'_{\underline{\beta}}),$$

where $\pi_{\underline{\beta}'} = \pi^{(0)}_{\underline{\beta}'} \in \widetilde{\mathcal{S}}_{(n,m)}$ is a constant dual d-spinor. The problem of definition of spinors and twistors on la-spaces was firstly considered in [Vacaru and Ostaf 1994] (see also [Vacaru 1987] and [Vacaru and Ostaf 1996b]) in connection with the possibility to extend the equations (1.66) and theirs solutions (1.67), by using nearly autoparallel maps, on curved, locally isotropic or anisotropic, spaces.

I.5.2 Mutual transforms of d-tensors and d-spinors.

The spinor algebra for spaces of higher dimensions can not be considered as a real alternative to the tensor algebra as for locally isotropic spaces of dimensions $n = 3,4$ [Penrose and Rindler 1984, 1986]. The same holds true for la-spaces and we emphasize that it is not quite convenient to perform a spinor calculus for dimensions $n,m >> 4$. Nevertheless, the concept of spinors is important for every type of spaces, we can deeply understand the fundamental properties of geometical objects on la-spaces, and we shall consider in this subsection some questions concerning transforms of d-tensor objects into d-spinor ones.

I.5.2.1 Transformation of d-tensors into d-spinors.

In order to pass from d-tensors to d-spinors we must use σ-objects (1.55) written in reduced or irreduced form (in dependence of fixed values of dimensions n and m):

$$(\sigma_{\widehat{\alpha}})^{\cdot\underline{\gamma}}_{\underline{\beta}},\ (\sigma^{\widehat{\alpha}})^{\underline{\beta\gamma}},\ (\sigma^{\widehat{\alpha}})_{\underline{\beta\gamma}}, ..., (\sigma_{\widehat{a}})^{\underline{bc}}, ..., (\sigma_{\widehat{i}})_{\underline{jk}}, ..., (\sigma_{\widehat{a}})^{AA'}, ..., (\sigma^{\widehat{i}})_{II'}, ... \tag{1.68}$$

It is obvious that contracting with corresponding σ-objects (1.68) we can introduce instead of d-tensors indices the d-spinor ones, for instance,

$$\omega^{\underline{\beta\gamma}} = (\sigma^{\widehat{\alpha}})^{\underline{\beta\gamma}}\omega_{\widehat{\alpha}},\quad \omega_{AB'} = (\sigma^{\widehat{a}})_{AB'}\omega_{\widehat{a}},\quad ..., \zeta^{\cdot\underline{i}}_{\underline{j}} = (\sigma^{\widehat{k}})^{\cdot\underline{i}}_{\underline{j}}\zeta_{\widehat{k}}, ...$$

For d-tensors containing groups of antisymmetric indices there is a more simple procedure of theirs transforming into d-spinors because the objects

$$(\sigma_{\widehat{\alpha\beta}...\widehat{\gamma}})^{\underline{\delta\nu}}, \quad (\sigma^{\widehat{ab}...\widehat{c}})^{\underline{de}}, \quad ..., (\sigma^{\widehat{ij}...\widehat{k}})_{II'}, \quad ... \tag{1.69}$$

can be used for sets of such indices into pairs of d-spinor indices. Let us enumerate some properties of σ-objects of type (1.69) (for simplicity we consider only h-components having q indices $\widehat{i}, \widehat{j}, \widehat{k}, ...$ taking values from 1 to n; the properties of v-components can be written in a similar manner with respect to indices $\widehat{a}, \widehat{b}, \widehat{c}...$ taking values from 1 to m):

$$(\sigma_{\widehat{i}...\widehat{j}})^{\underline{kl}} \text{ is } \begin{cases} \text{symmetric on } \underline{k}, \underline{l} & \text{for } n-2q \equiv 1,7 \ (mod\ 8); \\ \text{antisymmetric on } \underline{k}, \underline{l} & \text{for } n-2q \equiv 3,5 \ (mod\ 8) \end{cases} \tag{1.70}$$

for odd values of n, and an object

$$(\sigma_{\widehat{i}...\widehat{j}})^{IJ} \ \left((\sigma_{\widehat{i}...\widehat{j}})^{I'J'}\right) \tag{1.71}$$

$$\text{is } \begin{cases} \text{symmetric on } I, J \ (I', J') & \text{for } n-2q \equiv 0 \ (mod\ 8); \\ \text{antisymmetric on } I, J \ (I', J') & \text{for } n-2q \equiv 4 \ (mod\ 8) \end{cases}$$

or

$$(\sigma_{\widehat{i}...\widehat{j}})^{IJ'} = \pm(\sigma_{\widehat{i}...\widehat{j}})^{J'I} \begin{cases} n+2q \equiv 6 (mod 8); \\ n+2q \equiv 2 (mod 8), \end{cases} \tag{1.72}$$

with vanishing of the rest of reduced components of the d-tensor $(\sigma_{\widehat{i}...\widehat{j}})^{\underline{kl}}$ with prime/unprime sets of indices.

I.5.2.2 Transformation of d-spinors into d-tensors; fundamental d–spinors.

We can transform every d-spinor $\xi^{\underline{\alpha}} = \left(\xi^{\underline{i}}, \xi^{\underline{a}}\right)$ into a corresponding d-tensor. For simplicity, we consider this construction only for a h-component $\xi^{\underline{i}}$ on a h-space being of dimension n. The values

$$\xi^{\underline{\alpha}} \xi^{\underline{\beta}} (\sigma^{\widehat{i}...\widehat{j}})_{\underline{\alpha\beta}} \quad (n \text{ is odd}) \tag{1.73}$$

or

$$\xi^I \xi^J (\sigma^{\widehat{i}...\widehat{j}})_{IJ} \ \left(\text{or } \xi^{I'} \xi^{J'} (\sigma^{\widehat{i}...\widehat{j}})_{I'J'}\right) \ (n \text{ is even}) \tag{1.74}$$

with a different number of indices $\widehat{i}...\widehat{j}$, taken together, defines the h-spinor $\xi^{\underline{i}}$ to an accuracy to the sign. We emphasize that it is necessary to choose only those h-components of d-tensors (1.73) (or (1.74)) which are symmetric on pairs of indices $\underline{\alpha\beta}$ (or IJ (or $I'J'$)) and the number q of indices $\widehat{i}...\widehat{j}$ satisfies the condition (as a respective consequence of the properties (1.70) and/or (1.71), (1.72))

$$n - 2q \equiv 0, 1, 7 \ (mod\ 8). \tag{1.75}$$

Of special interest is the case when

$$q=\frac{1}{2}(n\pm 1)\ \ (n \text{ is odd}) \tag{1.76}$$

or

$$q=\frac{1}{2}n\ \ (n \text{ is even}). \tag{1.77}$$

If all expressions (1.73) and/or (1.74) are zero for all values of q with the exception of one or two ones defined by the condition (1.76) (or (1.77)), the value $\xi^{\widehat{i}}$ (or ξ^{I} $(\xi^{I'})$) is called a fundamental h-spinor. Defining in a similar manner the fundamental v-spinors we can introduce fundamental d-spinors as pairs of fundamental h- and v-spinors. Here we remark that a h(v)-spinor $\xi^{\widehat{i}}$ $(\xi^{\widehat{a}})$ (we can also consider reduced components) is always a fundamental one for $n(m)<7$, which is a consequence of (1.75)).

Finally, in this section, we note that the geometry of fundamental h- and v-spinors is similar to that of usual fundamental spinors (see Appendix to the monograph [Penrose and Rindler 1986]). We omit such details in this work, but emphasize that constructions with fundamental d-spinors, for a la-space, must be adapted to the corresponding global splitting by N-connection of the space.

I.6 The Differential Geometry of Locally Anisotropic Spinors

The goal of the section is to formulate the differential geometry of d-spinors for la-spaces.

We shall use denotations of type

$$v^{\alpha}=(v^{i},v^{a})\in\sigma^{\alpha}=(\sigma^{i},\sigma^{a})\ \text{ and } \zeta^{\underline{\alpha}}=(\zeta^{\underline{i}},\zeta^{\underline{a}})\in\sigma^{\underline{\alpha}}=(\sigma^{\underline{i}},\sigma^{\underline{a}})$$

for, respectively, elements of modules of d-vector and irreduced d-spinor fields (see details in [Vacaru 1996]). D–tensors and d–spinor tensors (irreduced or reduced) will be interpreted as elements of corresponding σ -modules, for instance,

$$q^{\alpha}{}_{\beta\ldots}\in\sigma^{\alpha}{}_{\beta\ldots},\psi^{\underline{\alpha}}{}_{\underline{\beta}}{}^{\underline{\gamma}}{}_{\ldots}\in\sigma^{\underline{\alpha}}{}_{\underline{\beta}}{}^{\underline{\gamma}}{}_{\ldots},\xi^{II'}{}_{JK'N'}\in\sigma^{II'}{}_{JK'N'},\ldots$$

We can establish a correspondence between the la-adapted metric $g_{\alpha\beta}$ (1.12) and d-spinor metric $\epsilon_{\underline{\alpha}\underline{\beta}}$ (ϵ-objects (1.60) for both h- and v-subspaces of $\mathcal{E}$,) of a la-space $\mathcal{E}$ by using the relation

$$g_{\alpha\beta}=-\frac{1}{N(n)+N(m)}((\sigma_{(\alpha}(u))^{\underline{\alpha}_1\underline{\beta}_1}(\sigma_{\beta)}(u))^{\underline{\beta}_2\underline{\alpha}_2})\epsilon_{\underline{\alpha}_1\underline{\alpha}_2}\epsilon_{\underline{\beta}_1\underline{\beta}_2}, \tag{1.78}$$

where

$$(\sigma_{\alpha}(u))^{\underline{\nu}\underline{\gamma}}=l^{\widehat{\alpha}}_{\alpha}(u)(\sigma_{\widehat{\alpha}})^{\underline{\nu}\underline{\gamma}}, \tag{1.79}$$

which is a consequence of formulas (1.52)-(1.57). In brief we can write (1.78) as

(1.80)
$$g_{\alpha\beta} = \epsilon_{\underline{\alpha}_1\underline{\alpha}_2}\epsilon_{\underline{\beta}_1\underline{\beta}_2}$$

if the σ-objects are considered as a fixed structure, whereas ϵ-objects are treated as caring the metric "dynamics " , on la-space. This variant is used, for instance, in the so-called 2-spinor geometry [Penrose and Rindler 1984, 1986] and should be preferred if we have to make explicit the algebraic symmetry properties of d-spinor objects. An alternative way is to considered as fixed the algebraic structure of ϵ-objects and to use variable components of σ-objects of type (1.79) for developing a variational d-spinor approach to gravitational and matter field interactions on la-spaces (the spinor Ashtekar variables [Ashtekar, Romano and Ranjet 1989] are introduced in this manner).

We note that a d-spinor metric

$$\epsilon_{\underline{\nu\tau}} = \begin{pmatrix} \epsilon_{\underline{ij}} & 0 \\ 0 & \epsilon_{\underline{ab}} \end{pmatrix}$$

on the d-spinor space $\mathcal{S} = (\mathcal{S}_{(h)}, \mathcal{S}_{(v)})$ can have symmetric or antisymmetric h (v) - components $\epsilon_{\underline{ij}}$ $(\epsilon_{\underline{ab}})$, see ϵ-objects (1.60). For simplicity, in this section (in order to avoid cumbersome calculations connected with eight-fold periodicity on dimensions n and m of a la-space $\mathcal{E}$) we shall develop a general d-spinor formalism only by using irreduced spinor spaces $\mathcal{S}_{(h)}$ and $\mathcal{S}_{(v)}$.

I.6.1 D-covariant derivation on la-spaces.

Let $\mathcal{E}$ be a la–space. We define the action on a d-spinor of a d-covariant operator

$$\nabla_\alpha = (\nabla_i, \nabla_a) = (\sigma_\alpha)^{\underline{\alpha}_1\underline{\alpha}_2}\nabla_{\underline{\alpha}_1\underline{\alpha}_2} = \left((\sigma_i)^{\underline{i}_1\underline{i}_2}\nabla_{\underline{i}_1\underline{i}_2},\ (\sigma_a)^{\underline{a}_1\underline{a}_2}\nabla_{\underline{a}_1\underline{a}_2}\right)$$

(in brief, we shall write

$$\nabla_\alpha = \nabla_{\underline{\alpha}_1\underline{\alpha}_2} = \left(\nabla_{\underline{i}_1\underline{i}_2},\ \nabla_{\underline{a}_1\underline{a}_2}\right))$$

as a map

$$\nabla_{\underline{\alpha}_1\underline{\alpha}_2}\ :\ \sigma^{\underline{\beta}} \to \sigma_\alpha^{\underline{\beta}} = \sigma_{\underline{\alpha}_1\underline{\alpha}_2}^{\underline{\beta}}$$

satisfying conditions

$$\nabla_\alpha(\xi^{\underline{\beta}} + \eta^{\underline{\beta}}) = \nabla_\alpha\xi^{\underline{\beta}} + \nabla_\alpha\eta^{\underline{\beta}},$$

and

$$\nabla_\alpha(f\xi^{\underline{\beta}}) = f\nabla_\alpha\xi^{\underline{\beta}} + \xi^{\underline{\beta}}\nabla_\alpha f$$

for every $\xi^{\underline{\beta}}, \eta^{\underline{\beta}} \in \sigma^{\underline{\beta}}$ and f being a scalar field on $\mathcal{E}$. It is also required that one holds the Leibnitz rule

$$(\nabla_\alpha\zeta_{\underline{\beta}})\eta^{\underline{\beta}} = \nabla_\alpha(\zeta_{\underline{\beta}}\eta^{\underline{\beta}}) - \zeta_{\underline{\beta}}\nabla_\alpha\eta^{\underline{\beta}}$$

and that ∇_α is a real operator, i.e. it commuters with the operation of complex conjugation:

$$\overline{\nabla_\alpha \psi_{\underline{\alpha}\underline{\beta}\underline{\gamma}...}} = \nabla_\alpha (\overline{\psi}_{\underline{\alpha}\underline{\beta}\underline{\gamma}...}).$$

Let now analyze the question on uniqueness of action on d-spinors of an operator ∇_α satisfying necessary conditions . Denoting by $\nabla_\alpha^{(1)}$ and ∇_α two such d-covariant operators we consider the map

$$(\nabla_\alpha^{(1)} - \nabla_\alpha) : \sigma^{\underline{\beta}} \to \sigma^{\underline{\beta}}_{\underline{\alpha}_1 \underline{\alpha}_2}. \tag{1.81}$$

Because the action on a scalar f of both operators $\nabla_\alpha^{(1)}$ and ∇_α must be identical, i.e.

$$\nabla_\alpha^{(1)} f = \nabla_\alpha f, \tag{1.82}$$

the action (1.81) on $f = \omega_{\underline{\beta}} \xi^{\underline{\beta}}$ must be written as

$$(\nabla_\alpha^{(1)} - \nabla_\alpha)(\omega_{\underline{\beta}} \xi^{\underline{\beta}}) = 0.$$

In consequence we conclude that there is an element $\Theta_{\underline{\alpha}_1 \underline{\alpha}_2 \underline{\beta}}{}^{\underline{\gamma}} \in \sigma_{\underline{\alpha}_1 \underline{\alpha}_2 \underline{\beta}}{}^{\underline{\gamma}}$ for which

$$\nabla^{(1)}_{\underline{\alpha}_1 \underline{\alpha}_2} \xi^{\underline{\gamma}} = \nabla_{\underline{\alpha}_1 \underline{\alpha}_2} \xi^{\underline{\gamma}} + \Theta_{\underline{\alpha}_1 \underline{\alpha}_2 \underline{\beta}}{}^{\underline{\gamma}} \xi^{\underline{\beta}} \text{ and } \nabla^{(1)}_{\underline{\alpha}_1 \underline{\alpha}_2} \omega_{\underline{\beta}} = \nabla_{\underline{\alpha}_1 \underline{\alpha}_2} \omega_{\underline{\beta}} - \Theta_{\underline{\alpha}_1 \underline{\alpha}_2 \underline{\beta}}{}^{\underline{\gamma}} \omega_{\underline{\gamma}} . \tag{1.83}$$

The action of the operator (1.81) on a d-vector $v^\beta = v^{\underline{\beta}_1 \underline{\beta}_2}$ can be written by using formula (1.83) for both indices $\underline{\beta}_1$ and $\underline{\beta}_2$:

$$(\nabla_\alpha^{(1)} - \nabla_\alpha) v^{\underline{\beta}_1 \underline{\beta}_2} = \Theta_{\alpha \underline{\gamma}}{}^{\underline{\beta}_1} v^{\underline{\gamma} \underline{\beta}_2} + \Theta_{\alpha \underline{\gamma}}{}^{\underline{\beta}_2} v^{\underline{\beta}_1 \underline{\gamma}} =$$

$$(\Theta_{\alpha \underline{\gamma}_1}{}^{\underline{\beta}_1} \delta_{\underline{\gamma}_2}{}^{\underline{\beta}_2} + \Theta_{\alpha \underline{\gamma}_1}{}^{\underline{\beta}_2} \delta_{\underline{\gamma}_2}{}^{\underline{\beta}_1}) v^{\underline{\gamma}_1 \underline{\gamma}_2} = Q^\beta{}_{\alpha\gamma} v^\gamma,$$

where

$$Q^\beta{}_{\alpha\gamma} = Q^{\underline{\beta}_1 \underline{\beta}_2}{}_{\underline{\alpha}_1 \underline{\alpha}_2 \ \underline{\gamma}_1 \underline{\gamma}_2} = \Theta_{\alpha \underline{\gamma}_1}{}^{\underline{\beta}_1} \delta_{\underline{\gamma}_2}{}^{\underline{\beta}_2} + \Theta_{\alpha \underline{\gamma}_1}{}^{\underline{\beta}_2} \delta_{\underline{\gamma}_2}{}^{\underline{\beta}_1}. \tag{1.84}$$

The d-commutator $\nabla_{[\alpha} \nabla_{\beta]}$ defines the d-torsion (see (1.27),(1.28) and (1.29)). So, applying operators $\nabla^{(1)}_{[\alpha} \nabla^{(1)}_{\beta]}$ and $\nabla_{[\alpha} \nabla_{\beta]}$ on $f = \omega_{\underline{\beta}} \xi^{\underline{\beta}}$ we can write

$$T^{(1)\gamma}{}_{\alpha\beta} - T^\gamma{}_{\alpha\beta} = Q^\gamma{}_{\beta\alpha} - Q^\gamma{}_{\alpha\beta}$$

with $Q^\gamma{}_{\alpha\beta}$ from (1.84).

The action of operator $\nabla_\alpha^{(1)}$ on d-spinor tensors of type $\chi_{\underline{\alpha}_1 \underline{\alpha}_2 \underline{\alpha}_3 ...}{}^{\underline{\beta}_1 \underline{\beta}_2 ...}$ must be constructed by using formula (1.83) for every upper index $\underline{\beta}_1 \underline{\beta}_2 ...$ and formula (1.84) for every lower index $\underline{\alpha}_1 \underline{\alpha}_2 \underline{\alpha}_3 ...$.

I.6.2 Infeld - van der Waerden coefficients and d-connections.

Let

$$\delta_{\underline{\alpha}}{}^{\underline{\alpha}} = \left(\delta_{\underline{\mathbf{1}}}{}^{\underline{i}}, \delta_{\underline{\mathbf{2}}}{}^{\underline{i}}, ..., \delta_{\underline{\mathbf{N}(n)}}{}^{\underline{i}}, \delta_{\underline{\mathbf{1}}}{}^{\underline{a}}, \delta_{\underline{\mathbf{2}}}{}^{\underline{a}}, ..., \delta_{\underline{\mathbf{N}(m)}}{}^{\underline{i}}\right)$$

be a d-spinor basis. The dual to it basis is denoted as

$$\delta_{\underline{\alpha}}{}^{\underline{\alpha}} = \left(\delta_{\underline{i}}{}^{\underline{\mathbf{1}}}, \delta_{\underline{i}}{}^{\underline{\mathbf{2}}}, ..., \delta_{\underline{i}}{}^{\underline{\mathbf{N}(n)}}, \delta_{\underline{i}}{}^{\underline{\mathbf{1}}}, \delta_{\underline{i}}{}^{\underline{\mathbf{2}}}, ..., \delta_{\underline{i}}{}^{\underline{\mathbf{N}(m)}}\right).$$

A d-spinor $\kappa^{\underline{\alpha}} \in \sigma^{\ \underline{\alpha}}$ has components $\kappa^{\underline{\alpha}} = \kappa^{\underline{\alpha}} \delta_{\underline{\alpha}}{}^{\underline{\alpha}}$. Taking into account that

$$\delta_{\underline{\alpha}}{}^{\underline{\alpha}} \delta_{\underline{\beta}}{}^{\underline{\beta}} \nabla_{\underline{\alpha}\underline{\beta}} = \nabla_{\underline{\alpha}\underline{\beta}},$$

we write out the components $\nabla_{\underline{\alpha}\underline{\beta}}\ \kappa^{\underline{\gamma}}$ as

(1.85) $$\delta_{\underline{\alpha}}{}^{\underline{\alpha}}\ \delta_{\underline{\beta}}{}^{\underline{\beta}}\ \delta_{\underline{\gamma}}{}^{\underline{\gamma}}\ \nabla_{\underline{\alpha}\underline{\beta}} \kappa^{\underline{\gamma}} = \delta_{\underline{\epsilon}}{}^{\underline{\tau}}\ \delta_{\underline{\tau}}{}^{\underline{\gamma}}\ \nabla_{\underline{\alpha}\underline{\beta}} \kappa^{\underline{\epsilon}} + \kappa^{\underline{\epsilon}}\ \delta_{\underline{\epsilon}}{}^{\underline{\gamma}}\ \nabla_{\underline{\alpha}\underline{\beta}} \delta_{\underline{\epsilon}}{}^{\underline{\epsilon}} = \nabla_{\underline{\alpha}\underline{\beta}} \kappa^{\underline{\gamma}} + \kappa^{\underline{\epsilon}} \gamma^{\underline{\gamma}}{}_{\underline{\alpha}\underline{\beta}\underline{\epsilon}},$$

where the coordinate components of the d-spinor connection $\gamma^{\underline{\gamma}}{}_{\underline{\alpha}\underline{\beta}\underline{\epsilon}}$ are defined as

(1.86) $$\gamma^{\underline{\gamma}}{}_{\underline{\alpha}\underline{\beta}\underline{\epsilon}} \doteq \delta_{\underline{\tau}}{}^{\underline{\gamma}}\ \nabla_{\underline{\alpha}\underline{\beta}} \delta_{\underline{\epsilon}}{}^{\underline{\tau}}.$$

We call the Infeld - van der Waerden d-symbols a set of σ–objects $(\sigma_\alpha)^{\underline{\alpha}\underline{\beta}}$ parametrized with respect to a coordinate d-spinor basis. Defining $\nabla_\alpha = (\sigma_\alpha)^{\underline{\alpha}\underline{\beta}}\ \nabla_{\underline{\alpha}\underline{\beta}}$, introducing denotations $\gamma^{\underline{\gamma}}{}_{\alpha\underline{\tau}} \doteq \gamma^{\underline{\gamma}}{}_{\underline{\alpha}\underline{\beta}\underline{\tau}} (\sigma_\alpha)^{\underline{\alpha}\underline{\beta}}$ and using properties (1.85) we can write relations

(1.87) $$l^\alpha_\alpha\ \delta_{\underline{\beta}}{}^{\underline{\beta}}\ \nabla_\alpha \kappa^{\underline{\beta}} = \nabla_\alpha \kappa^{\underline{\beta}} + \kappa^{\underline{\delta}}\ \gamma^{\underline{\beta}}{}_{\alpha\underline{\delta}}$$

and

(1.88) $$l^\alpha_\alpha\ \delta_{\underline{\beta}}{}^{\underline{\beta}}\ \nabla_\alpha\ \mu_{\underline{\beta}} = \nabla_\alpha\ \mu_{\underline{\beta}} - \mu_{\underline{\delta}} \gamma^{\underline{\delta}}{}_{\alpha\underline{\beta}}$$

for d-covariant derivations $\nabla_{\underline{\alpha}} \kappa^{\underline{\beta}}$ and $\nabla_{\underline{\alpha}}\ \mu_{\underline{\beta}}$.

We can consider expressions similar to (1.87) and (1.88) for values having both types of d-spinor and d-tensor indices, for instance,

$$l^\alpha_\alpha\ l^\gamma_\gamma\ \delta_{\underline{\delta}}{}^{\underline{\delta}}\ \nabla_\alpha \theta_{\underline{\delta}}{}^{\gamma} = \nabla_\alpha \theta_{\underline{\delta}}{}^{\gamma} - \theta_{\underline{\epsilon}}{}^{\gamma} \gamma^{\underline{\epsilon}}{}_{\alpha\underline{\delta}} + \theta_{\underline{\delta}}{}^{\tau}\ \Gamma^{\ \gamma}{}_{\alpha\tau}$$

(we can prove this by a straightforward calculation of the derivation $\nabla_\alpha (\theta_{\underline{\epsilon}}{}^{\tau}\ \ \delta_{\underline{\delta}}{}^{\underline{\epsilon}}\ l^\gamma_\tau))$.

Now we shall consider some possible relations between components of d-connections $\gamma^{\underline{\epsilon}}{}_{\alpha\underline{\delta}}$ and $\Gamma^{\ \gamma}{}_{\alpha\tau}$ and derivations of $(\sigma_\alpha)^{\underline{\alpha}\underline{\beta}}$. According to definitions (1.12) we can write

$$\Gamma^\alpha{}_{\beta\gamma} = l^\alpha_\alpha \nabla_\gamma l^\alpha_\beta = l^\alpha_\alpha \nabla_\gamma (\sigma_\beta)^{\underline{\epsilon}\underline{\tau}} = l^\alpha_\alpha \nabla_\gamma ((\sigma_\beta)^{\underline{\epsilon}\underline{\tau}} \delta_{\underline{\epsilon}}{}^{\underline{\epsilon}} \delta_{\underline{\tau}}{}^{\underline{\tau}}) =$$

$$l^{\alpha}_{\alpha}\delta_{\underline{\alpha}}^{\ \underline{\alpha}}\delta_{\underline{\epsilon}}^{\ \epsilon}\nabla_{\gamma}(\sigma_{\beta})^{\underline{\alpha\epsilon}} + l^{\alpha}_{\alpha}(\sigma_{\beta})^{\underline{\epsilon\tau}}(\delta_{\underline{\tau}}^{\ \tau}\nabla_{\gamma}\delta_{\underline{\epsilon}}^{\ \epsilon} + \delta_{\underline{\epsilon}}^{\ \epsilon}\nabla_{\gamma}\delta_{\underline{\tau}}^{\ \tau}) =$$

$$l^{\alpha}_{\underline{\epsilon\tau}}\ \nabla_{\gamma}(\sigma_{\beta})^{\underline{\epsilon\tau}} + l^{\alpha}_{\underline{\mu\nu}}\delta_{\underline{\epsilon}}^{\ \underline{\mu}}\delta_{\underline{\tau}}^{\ \underline{\nu}}(\sigma_{\beta})^{\underline{\epsilon\tau}}(\delta_{\underline{\tau}}^{\ \tau}\nabla_{\gamma}\delta_{\underline{\epsilon}}^{\ \epsilon} + \delta_{\underline{\epsilon}}^{\ \epsilon}\nabla_{\gamma}\delta_{\underline{\tau}}^{\ \tau}),$$

where $l^{\alpha}_{\alpha} = (\sigma_{\underline{\epsilon\tau}})^{\alpha}$, from which it follows

$$(\sigma_{\alpha})^{\underline{\mu\nu}}(\sigma_{\underline{\alpha\beta}})^{\beta}\Gamma^{\alpha}_{\ \gamma\beta} = (\sigma_{\underline{\alpha\beta}})^{\beta}\nabla_{\gamma}(\sigma_{\alpha})^{\underline{\mu\nu}} + \delta_{\underline{\beta}}^{\ \underline{\nu}}\gamma^{\underline{\mu}}_{\ \gamma\underline{\alpha}} + \delta_{\underline{\alpha}}^{\ \underline{\mu}}\gamma^{\underline{\nu}}_{\ \gamma\underline{\beta}}.$$

Connecting the last expression on $\underline{\beta}$ and $\underline{\nu}$ and using an orthonormalized d–spinor basis when $\gamma^{\underline{\beta}}_{\ \gamma\underline{\beta}} = 0$ (a consequence from (1.86)) we have

$$\gamma^{\underline{\mu}}_{\ \gamma\underline{\alpha}} = \frac{1}{N(n)+N(m)}(\Gamma^{\underline{\mu\beta}}_{\ \ \ \gamma\ \underline{\alpha\beta}} - (\sigma_{\underline{\alpha\beta}})^{\beta}\nabla_{\gamma}(\sigma_{\beta})^{\underline{\mu\beta}}), \tag{1.89}$$

where

$$\Gamma^{\underline{\mu\beta}}_{\ \ \ \gamma\ \underline{\alpha\beta}} = (\sigma_{\alpha})^{\underline{\mu\beta}}(\sigma_{\underline{\alpha\beta}})^{\beta}\Gamma^{\alpha}_{\ \gamma\beta}. \tag{1.90}$$

We also note here that, for instance, for the canonical and Berwald connections, Christoffel d–symbols we can express the d-spinor connection (1.90) through corresponding locally adapted derivations of components of metric and N-connection by introducing respectively coefficients (1.22) and (1.20), or (1.23) instead of $\Gamma^{\alpha}_{\gamma\beta}$ in (1.90) and than in (1.89).

I.6.3 D-spinors of la-space curvature and torsion.

The d-tensor indices of the commutator (1.35), $\Delta_{\alpha\beta}$, can be transformed into d-spinor ones:

$$\Box_{\underline{\alpha\beta}} = (\sigma^{\alpha\beta})_{\underline{\alpha\beta}}\Delta_{\alpha\beta} = (\Box_{\underline{ij}}, \Box_{\underline{ab}}), \tag{1.91}$$

with h- and v-components,

$$\Box_{\underline{ij}} = (\sigma^{\alpha\beta})_{\underline{ij}}\Delta_{\alpha\beta} \text{ and } \Box_{\underline{ab}} = (\sigma^{\alpha\beta})_{\underline{ab}}\Delta_{\alpha\beta},$$

being symmetric or antisymmetric in dependence of corresponding values of dimensions n and m (see eight-fold parametizations (1.69), (1.70) and (1.71)). Considering the actions of operator (1.91) on d-spinors $\pi^{\underline{\gamma}}$ and $\mu_{\underline{\gamma}}$ we introduce the d-spinor curvature $X_{\underline{\delta}\ \ \underline{\alpha\beta}}^{\ \ \underline{\gamma}}$ as to satisfy equations

$$\Box_{\underline{\alpha\beta}}\ \pi^{\underline{\gamma}} = X_{\underline{\delta}\ \ \underline{\alpha\beta}}^{\ \ \underline{\gamma}}\pi^{\underline{\delta}} \text{ and } \Box_{\underline{\alpha\beta}}\ \mu_{\underline{\gamma}} = X_{\underline{\gamma}\ \ \underline{\alpha\beta}}^{\ \ \underline{\delta}}\mu_{\underline{\delta}}. \tag{1.92}$$

The gravitational d-spinor $\Psi_{\underline{\alpha\beta\gamma\delta}}$ is defined by a corresponding symmetrization of d-spinor indices:

$$\Psi_{\underline{\alpha\beta\gamma\delta}} = X_{(\underline{\alpha}|\underline{\beta}|\underline{\gamma\delta})}.$$

We note that d-spinor tensors $X_{\underline{\delta}}\ {}^{\underline{\gamma}}{}_{\underline{\alpha\beta}}$ and $\Psi_{\underline{\alpha\beta\gamma\delta}}$ are transformed into similar 2-spinor objects on locally isotropic spaces [Penrose and Rindler 1984, 1986] if we consider vanishing of the N–connection structure and a limit to a locally isotropic space.

Putting $\delta_{\underline{\gamma}}\ {}^{\underline{\gamma}}$ instead of $\mu_{\underline{\gamma}}$ in (1.92) and using (1.93) we can express respectively the curvature and gravitational d-spinors as

$$X_{\underline{\gamma\delta\alpha\beta}} = \delta_{\underline{\delta\tau}}\Box_{\underline{\alpha\beta}}\delta_{\underline{\gamma}}\ {}^{\underline{\tau}} \text{ and } \Psi_{\underline{\gamma\delta\alpha\beta}} = \delta_{\underline{\delta\tau}}\Box_{(\underline{\alpha\beta}}\delta_{\underline{\gamma})}\ {}^{\underline{\tau}}.$$

The d-spinor torsion $T^{\underline{\gamma}_1\underline{\gamma}_2}{}_{\underline{\alpha\beta}}$ is defined similarly as for d-tensors (see (1.36)) by using the d-spinor commutator (1.91) and equations

$$\Box_{\underline{\alpha\beta}} f = T^{\underline{\gamma}_1\underline{\gamma}_2}{}_{\underline{\alpha\beta}}\nabla_{\underline{\gamma}_1\underline{\gamma}_2} f. \tag{1.93}$$

The d-spinor components $R_{\underline{\gamma}_1\underline{\gamma}_2}\ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha\beta}}$ of the curvature d-tensor $R_{\gamma}\ {}^{\delta}{}_{\alpha\beta}$ can be computed by using relations (1.90), and (1.91) and (1.93) as to satisfy the equations (the d-spinor analogous of equations (1.37))

$$(\Box_{\underline{\alpha\beta}} - T^{\underline{\gamma}_1\underline{\gamma}_2}{}_{\underline{\alpha\beta}}\nabla_{\underline{\gamma}_1\underline{\gamma}_2})V^{\underline{\delta}_1\underline{\delta}_2} = R_{\underline{\gamma}_1\underline{\gamma}_2}\ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha\beta}}V^{\underline{\gamma}_1\underline{\gamma}_2}, \tag{1.94}$$

here d-vector $V^{\underline{\gamma}_1\underline{\gamma}_2}$ is considered as a product of d-spinors, i.e. $V^{\underline{\gamma}_1\underline{\gamma}_2} = \nu^{\underline{\gamma}_1}\mu^{\underline{\gamma}_2}$. We find

$$R_{\underline{\gamma}_1\underline{\gamma}_2}\ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha\beta}} = \left(X_{\underline{\gamma}_1}\ {}^{\underline{\delta}_1}{}_{\underline{\alpha\beta}} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha\beta}}\quad \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1}\right)\delta_{\underline{\gamma}_2}\ {}^{\underline{\delta}_2} + \left(X_{\underline{\gamma}_2}\ {}^{\underline{\delta}_2}{}_{\underline{\alpha\beta}} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha\beta}}\quad \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2}\right)\delta_{\underline{\gamma}_1}\ {}^{\underline{\delta}_1}. \tag{1.95}$$

It is convenient to use this d-spinor expression for the curvature d-tensor

$$R_{\underline{\gamma}_1\underline{\gamma}_2}\ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2} = \left(X_{\underline{\gamma}_1}\ {}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2}\ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1}\right)\delta_{\underline{\gamma}_2}\ {}^{\underline{\delta}_2} + \left(X_{\underline{\gamma}_2}\ {}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\beta}_1\underline{\beta}_2}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2}\right)\delta_{\underline{\gamma}_1}\ {}^{\underline{\delta}_1}$$

in order to get the d-spinor components of the Ricci d-tensor

$$R_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} = R_{\underline{\gamma}_1\underline{\gamma}_2}\ {}^{\underline{\delta}_1\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\delta}_2} = X_{\underline{\gamma}_1}\ {}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2}\ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1} + X_{\underline{\gamma}_2}\ {}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\gamma}_1\underline{\delta}_2}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2} \tag{1.96}$$

and this d-spinor decomposition of the scalar curvature:

$$q\overleftarrow{R} = R^{\underline{\alpha}_1\underline{\alpha}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2} = X^{\underline{\alpha}_1\underline{\delta}_1}{}_{\underline{\alpha}_1}\ {}^{\underline{\alpha}_2}{}_{\underline{\delta}_1\underline{\alpha}_2} + T^{\underline{\tau}_1\underline{\tau}_2\underline{\alpha}_1}{}_{\underline{\alpha}_2\underline{\delta}_1}\ {}^{\underline{\alpha}_2}\ \gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\alpha}_1} + X^{\underline{\alpha}_2\underline{\delta}_2\underline{\alpha}_1}{}_{\underline{\alpha}_2\underline{\delta}_2\underline{\alpha}_1} + T^{\underline{\tau}_1\underline{\tau}_2}\ {}^{\underline{\alpha}_2\underline{\alpha}_1}{}_{\underline{\alpha}_1\ \underline{\delta}_2}\ \gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\alpha}_2}. \tag{1.97}$$

Putting (1.96) and (1.97) into (1.43) and, correspondingly, (1.41) we find the d-spinor components of the Einstein and $\Phi_{\alpha\beta}$ d-tensors:

$$\overleftarrow{G}_{\gamma\alpha} = \overleftarrow{G}_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} = X_{\underline{\gamma}_1}{}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2}\,\gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1} + X_{\underline{\gamma}_2}{}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\gamma}_1\underline{\delta}_2}\,\gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2} - \frac{1}{2}\varepsilon_{\underline{\gamma}_1\underline{\alpha}_1}\varepsilon_{\underline{\gamma}_2\underline{\alpha}_2}[X^{\underline{\beta}_1\underline{\mu}_1}{}_{\underline{\beta}_1}{}^{\underline{\beta}_2}{}_{\underline{\mu}_1\underline{\beta}_2} + T^{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1}{}_{\underline{\beta}_2\underline{\mu}_1}{}^{\underline{\beta}_2}\,\gamma^{\underline{\mu}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1} + X^{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1}{}_{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\beta}_1}{}^{\underline{\beta}_2\underline{\beta}_1}{}_{\underline{\delta}_2}\,\gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_2}] \tag{1.98}$$

and

$$\Phi_{\gamma\alpha} = \Phi_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2} = \frac{1}{2(n+m)}\varepsilon_{\underline{\gamma}_1\underline{\alpha}_1}\varepsilon_{\underline{\gamma}_2\underline{\alpha}_2}[X^{\underline{\beta}_1\underline{\mu}_1}{}_{\underline{\beta}_1}{}^{\underline{\beta}_2}{}_{\underline{\mu}_1\underline{\beta}_2} + T^{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1}{}_{\underline{\beta}_2\underline{\mu}_1}{}^{\underline{\beta}_2}\,\gamma^{\underline{\mu}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_1} + X^{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1}{}_{\underline{\beta}_2\underline{\mu}_2\underline{\nu}_1} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\beta}_1}{}^{\underline{\beta}_2\underline{\beta}_1}{}_{\underline{\delta}_2}\,\gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\beta}_2}] - \frac{1}{2}[X_{\underline{\gamma}_1}{}^{\underline{\delta}_1}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2}\,\gamma^{\underline{\delta}_1}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_1} + X_{\underline{\gamma}_2}{}^{\underline{\delta}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\delta}_1\underline{\gamma}_2} + T^{\underline{\tau}_1\underline{\tau}_2}{}_{\underline{\alpha}_1\underline{\alpha}_2\underline{\gamma}_1\underline{\delta}_2}\,\gamma^{\underline{\delta}_2}{}_{\underline{\tau}_1\underline{\tau}_2\underline{\gamma}_2}]. \tag{1.99}$$

The components of the conformal Weyl d-spinor can be computed by putting d-spinor values of the curvature (1.95) and Ricci (1.96) d-tensors into corresponding expression for the d-tensor (1.40). We omit this calculus in this work.

I.7 Field Equations on Locally Anisotropic Spaces

The problem of formulation gravitational and gauge field equations on different types of la–spaces is considered, for instance, in [Miron and Anastasiei 1994], [Bejancu 1990], [Asanov and Ponomarenko 1988] and [Vacaru and Goncharenko 1995]. In this section we shall introduce the basic field equations for gravitational and matter field la-interactions in a generalized form for generic la-spaces.

I.7.1 Locally anisotropic scalar field interactions.

Let $\varphi(u) = (\varphi_1(u), \varphi_2(u), ..., \varphi_k(u))$ be a complex k-component scalar field of mass μ on la-space $\mathcal{E}$. The d-covariant generalization of the conformally invariant (in the massless case) scalar field equation [Penrose and Rindler 1984, 1986] can be defined by using the d'Alambert locally anisotropic operator $\square = D^\alpha D_\alpha$, where D_α is a d-covariant derivation on $\mathcal{E}$ satisfying conditions (1.14) and (1.15):

$$(\square + \frac{n+m-2}{4(n+m-1)}\overleftarrow{R} + \mu^2)\varphi(u) = 0. \tag{1.100}$$

We must change d-covariant derivation D_α into ${}^\diamond D_\alpha = D_\alpha + ieA_\alpha$ and take into account the d-vector current

$$J_\alpha^{(0)}(u) = i((\overline{\varphi}(u) D_\alpha \varphi(u) - D_\alpha \overline{\varphi}(u))\varphi(u))$$

if interactions between locally anisotropic electromagnetic field (d-vector potential A_α), where e is the electromagnetic constant, and charged scalar field φ are considered. The equations (1.100) are (locally adapted to the N-connection structure)

Euler equations for the Lagrangian

(1.101)
$$\mathcal{L}^{(0)}(u) = \sqrt{|g|}\left[g^{\alpha\beta}\delta_{\alpha}\overline{\varphi}(u)\,\delta_{\beta}\varphi(u) - \left(\mu^2 + \frac{n+m-2}{4(n+m-1)}\right)\overline{\varphi}(u)\,\varphi(u)\right],$$

where $|g| = detg_{\alpha\beta}$.

The locally adapted variations of the action with Lagrangian (1.101) on variables $\varphi(u)$ and $\overline{\varphi}(u)$ leads to the locally anisotropic generalization of the energy-momentum tensor,

(1.102)
$$E^{(0,can)}_{\alpha\beta}(u) = \delta_{\alpha}\overline{\varphi}(u)\,\delta_{\beta}\varphi(u) + \delta_{\beta}\overline{\varphi}(u)\,\delta_{\alpha}\varphi(u) - \frac{1}{\sqrt{|g|}}g_{\alpha\beta}\mathcal{L}^{(0)}(u),$$

and a similar variation on the components of a d-metric (1.12) leads to a symmetric metric energy-momentum d-tensor,

(1.103)
$$E^{(0)}_{\alpha\beta}(u) = E^{(0,can)}_{(\alpha\beta)}(u) - \frac{n+m-2}{2(n+m-1)}\left[R_{(\alpha\beta)} + D_{(\alpha}D_{\beta)} - g_{\alpha\beta}\Box\right]\overline{\varphi}(u)\,\varphi(u).$$

Here we note that we can obtain a nonsymmetric energy-momentum d-tensor if we firstly vary on $G_{\alpha\beta}$ and than impose constraints of type (1.10) in order to have a compatibility with the N-connection structure. We also conclude that the existence of a N-connection in v-bundle $\mathcal{E}$ results in a nonequivalence of energy-momentum d-tensors (1.102) and (1.103), nonsymmetry of the Ricci tensor (see (1.33)), nonvanishing of the d-covariant derivation of the Einstein d-tensor (1.43), $D_{\alpha}\overleftarrow{G}^{\alpha\beta} \neq 0$ and, in consequence, a corresponding breaking of conservation laws on la-spaces when $D_{\alpha}E^{\alpha\beta} \neq 0$ [Miron and Anastasiei 1987, 1994]. The problem of formulation of conservation laws on la-spaces is discussed in details and two variants of its solution (by using nearly autoparallel maps and tensor integral formalism on la-multispaces) are proposed in [Vacaru and Ostaf 1994, 1996a] (see Chapter 3). In this section we shall present only straightforward generalizations of field equations and necessary formulas for energy-momentum d-tensors of matter fields on $\mathcal{E}$ considering that it is naturally that the conservation laws (usually being consequences of global, local and/or intrinsic symmetries of the fundamental space-time and of the type of field interactions) have to be broken on locally anisotropic spaces.

I.7.2 Proca equations on la–spaces.

Let consider a d-vector field $\varphi_{\alpha}(u)$ with mass μ^2 (locally anisotropic Proca field) interacting with exterior la-gravitational field. From the Lagrangian

(1.104)
$$\mathcal{L}^{(1)}(u) = \sqrt{|g|}\left[-\frac{1}{2}\overline{f}_{\alpha\beta}(u)\,f^{\alpha\beta}(u) + \mu^2\overline{\varphi}_{\alpha}(u)\,\varphi^{\alpha}(u)\right],$$

where $f_{\alpha\beta} = D_{\alpha}\varphi_{\beta} - D_{\beta}\varphi_{\alpha}$, one follows the Proca equations on la-spaces

(1.105)
$$D_{\alpha}f^{\alpha\beta}(u) + \mu^2\varphi^{\beta}(u) = 0.$$

Equations (1.105) are a first type constraints for $\beta = 0$. Acting with D_α on (1.105), for $\mu \neq 0$ we obtain second type constraints

$$D_\alpha \varphi^\alpha (u) = 0. \tag{1.106}$$

Putting (1.106) into (1.105) we obtain second order field equations with respect to φ_α :

$$\Box \varphi_\alpha (u) + R_{\alpha\beta} \varphi^\beta (u) + \mu^2 \varphi_\alpha (u) = 0. \tag{1.107}$$

The energy-momentum d-tensor and d-vector current following from the (1.107) can be written as

$$E^{(1)}_{\alpha\beta} (u) = -g^{\varepsilon\tau} \left(\overline{f}_{\beta\tau} f_{\alpha\varepsilon} + \overline{f}_{\alpha\varepsilon} f_{\beta\tau} \right) + \mu^2 \left(\overline{\varphi}_\alpha \varphi_\beta + \overline{\varphi}_\beta \varphi_\alpha \right) - \frac{g_{\alpha\beta}}{\sqrt{|g|}} \mathcal{L}^{(1)} (u)$$

and

$$J^{(1)}_\alpha (u) = i \left(\overline{f}_{\alpha\beta} (u) \varphi^\beta (u) - \overline{\varphi}^\beta (u) f_{\alpha\beta} (u) \right).$$

For $\mu = 0$ the d-tensor $f_{\alpha\beta}$ and the Lagrangian (1.104) are invariant with respect to locally anisotropic gauge transforms of type

$$\varphi_\alpha (u) \rightarrow \varphi_\alpha (u) + \delta_\alpha \Lambda (u),$$

where $\Lambda (u)$ is a d-differentiable scalar function, and we obtain a locally anisotropic variant of Maxwell theory.

I.7.3 La-gravitons on la-backgrounds.

Let a massless d-tensor field $h_{\alpha\beta} (u)$ is interpreted as a small perturbation of the locally anisotropic background metric d-field $g_{\alpha\beta} (u)$. Considering, for simplicity, a torsionless background we have locally anisotropic Fierz–Pauli equations

$$\Box h_{\alpha\beta} (u) + 2 R_{\tau\alpha\beta\nu} (u) \; h^{\tau\nu} (u) = 0 \tag{1.108}$$

and d-gauge conditions

$$D_\alpha h^\alpha_\beta (u) = 0, \quad h (u) \equiv h^\alpha_\beta (u) = 0, \tag{1.109}$$

where $R_{\tau\alpha\beta\nu} (u)$ is curvature d-tensor of the la-background space (these formulae can be obtained by using a perturbation formalism with respect to $h_{\alpha\beta} (u)$).

We note that we can rewrite d-tensor formulas (1.100)-(1.109) into similar d-spinor ones by using formulas (1.78)-(1.80), (1.90), (1.92) and (1.96)-(1.105) (for simplicity, we omit these considerations in this work).

I.7.4 Locally anisotropic Dirac equations.

Let denote the Dirac d-spinor field on $\mathcal{E}$ as $\psi(u) = (\psi^{\underline{\alpha}}(u))$ and consider as the generalized Lorentz transforms the group of automorphysm of the metric $G_{\widehat{\alpha}\widehat{\beta}}$ (see the la-frame decomposition of d-metric (1.54)).The d-covariant derivation of field $\psi(u)$ is written as

$$\overrightarrow{\nabla_\alpha}\psi = \left[\delta_\alpha + \frac{1}{4} C_{\widehat{\alpha}\widehat{\beta}\widehat{\gamma}}(u)\ \ l_\alpha^{\widehat{\alpha}}(u)\,\sigma^{\widehat{\beta}}\sigma^{\widehat{\gamma}}\right]\psi, \tag{1.110}$$

where coefficients $C_{\widehat{\alpha}\widehat{\beta}\widehat{\gamma}} = \left(D_\gamma l_{\widehat{\alpha}}^{\alpha}\right) l_{\widehat{\beta}\alpha}\, l_{\widehat{\gamma}}^{\gamma}$ generalize for la-spaces the corresponding Ricci coefficients on Riemannian spaces. Using σ-objects $\sigma^\alpha(u)$ (see (1.79) and (1.55)) we define the Dirac equations on la-spaces:

$$(i\sigma^\alpha(u)\,\overrightarrow{\nabla_\alpha} - \mu)\psi = 0, \tag{1.111}$$

which are the Euler equations for the Lagrangian

$$\mathcal{L}^{(1/2)}(u) = \sqrt{|g|}\{[\psi^+(u)\,\sigma^\alpha(u)\,\overrightarrow{\nabla_\alpha}\psi(u) - (\overrightarrow{\nabla_\alpha}\psi^+(u))\sigma^\alpha(u)\,\psi(u)] - \mu\psi^+(u)\,\psi(u)\}, \tag{1.112}$$

where $\psi^+(u)$ is the complex conjugation and transposition of the column $\psi(u)$.

From (1.112) we obtain the d-metric energy-momentum d-tensor

$$E_{\alpha\beta}^{(1/2)}(u) = \frac{i}{4}[\psi^+(u)\,\sigma_\alpha(u)\,\overrightarrow{\nabla_\beta}\psi(u) + \psi^+(u)\,\sigma_\beta(u)\,\overrightarrow{\nabla_\alpha}\psi(u) - (\overrightarrow{\nabla_\alpha}\psi^+(u))\sigma_\beta(u)\,\psi(u) - (\overrightarrow{\nabla_\beta}\psi^+(u))\sigma_\alpha(u)\,\psi(u)]$$

and the d-vector source

$$J_\alpha^{(1/2)}(u) = \psi^+(u)\,\sigma_\alpha(u)\,\psi(u).$$

We emphasize that la-interactions with exterior gauge fields can be introduced by changing the la-partial derivation from (1.110) in this manner:

$$\delta_\alpha \to \delta_\alpha + ie^\star B_\alpha, \tag{1.113}$$

where $e^\star$ and B_α are respectively the constant d-vector potential of la-gauge interactions on la-spaces (see [Vacaru and Goncharenko 1995] and the next subsection).

I.7.5 D-spinor Yang-Mills equations.

We consider a v-bundle $\mathcal{B}_E,\ \pi_B : \mathcal{B} \to E$, on la-space $\mathcal{E}$. Additionally to d-tensor and d-spinor indices we shall use capital Greek letters, $\Phi, \Upsilon, \Xi, \Psi, ...$ for fibre (of this bundle) indices (see details in [Penrose and Rindler 1984, 1986] for the case when the base space of the v-bundle π_B is a locally isotropic space–time). Let $\underline{\nabla}_\alpha$ be, for simplicity, a torsionless, linear connection in $\mathcal{B}_E$ satisfying conditions:

$$\underline{\nabla}_\alpha : \Upsilon^\Theta \to \Upsilon_\alpha^\Theta \quad \left[\text{ or } \Xi^\Theta \to \Xi_\alpha^\Theta\right],$$

$$\underline{\nabla}_{\alpha}\left(\lambda^{\Theta}+\nu^{\Theta}\right)=\underline{\nabla}_{\alpha}\lambda^{\Theta}+\underline{\nabla}_{\alpha}\nu^{\Theta},$$

$$\underline{\nabla}_{\alpha}\left(f\lambda^{\Theta}\right)=\lambda^{\Theta}\underline{\nabla}_{\alpha}f+f\underline{\nabla}_{\alpha}\lambda^{\Theta},\quad f\in\Upsilon^{\Theta}\ [\text{ or } \Xi^{\Theta}],$$

where by Υ^{Θ} $\left(\Xi^{\Theta}\right)$ we denote the module of sections of the real (complex) v-bundle $\mathcal{B}_E$ provided with the abstract index Θ. The curvature of connection $\underline{\nabla}_{\alpha}$ is defined as

$$K_{\alpha\beta\Omega}{}^{\Theta}\lambda^{\Omega}=\left(\underline{\nabla}_{\alpha}\underline{\nabla}_{\beta}-\underline{\nabla}_{\beta}\underline{\nabla}_{\alpha}\right)\lambda^{\Theta}.$$

For Yang-Mills fields as a rule one considers that $\mathcal{B}_E$ is enabled with a unitary (complex) structure (complex conjugation changes mutually the upper and lower Greek indices). It is useful to introduce instead of $K_{\alpha\beta\Omega}{}^{\Theta}$ a Hermitian matrix $F_{\alpha\beta\Omega}{}^{\Theta}=i\,K_{\alpha\beta\Omega}{}^{\Theta}$ connected with components of the Yang-Mills d-vector potential $B_{\alpha\Xi}{}^{\Phi}$ according the formula:

$$\frac{1}{2}F_{\alpha\beta\Xi}{}^{\Phi}=\underline{\nabla}_{[\alpha}B_{\beta]\Xi}{}^{\Phi}-iB_{[\alpha|\Lambda|}{}^{\Phi}B_{\beta]\Xi}{}^{\Lambda}, \tag{1.114}$$

where the la-space indices commute with capital Greek indices. The gauge transforms are written in the form:

$$B_{\alpha\Theta}{}^{\Phi}\mapsto B_{\alpha\widehat{\Theta}}{}^{\widehat{\Phi}}=B_{\alpha\Theta}{}^{\Phi}\ s_{\Phi}{}^{\widehat{\Phi}}\ q_{\widehat{\Theta}}{}^{\Theta}+is_{\Theta}{}^{\widehat{\Phi}}\underline{\nabla}_{\alpha}\ q_{\widehat{\Theta}}{}^{\Theta},$$

$$F_{\alpha\beta\Xi}{}^{\Phi}\mapsto F_{\alpha\beta\widehat{\Xi}}{}^{\widehat{\Phi}}=F_{\alpha\beta\Xi}{}^{\Phi}s_{\Phi}{}^{\widehat{\Phi}}q_{\widehat{\Xi}}{}^{\Xi},$$

where matrices $s_{\Phi}{}^{\widehat{\Phi}}$ and $q_{\widehat{\Xi}}{}^{\Xi}$ are mutually inverse (Hermitian conjugated in the unitary case). The Yang-Mills equations on la–spaces (see details in the next Chapter) are written in this form:

$$\underline{\nabla}^{\alpha}F_{\alpha\beta\Theta}{}^{\Psi}=J_{\beta\ \Theta}{}^{\Psi}, \tag{1.115}$$

$$\underline{\nabla}_{[\alpha}F_{\beta\gamma]\Theta}{}^{\Xi}=0. \tag{1.116}$$

We must introduce deformations of connection of type (1.14) and (1.15), $\underline{\nabla}_{\alpha}^{\star}\longrightarrow\underline{\nabla}_{\alpha}+P_{\alpha}$, (the deformation d-tensor P_{α} is induced by the torsion in v-bundle $\mathcal{B}_E$) into the definition of the curvature of la-gauge fields (1.114) and motion equations (1.115) and (1.116) if interactions are modeled on a generic la-space.

Now we can write out the field equations of the Einstein-Cartan theory in the d-spinor form. So, for the Einstein equations (1.42) we have

$$\overleftarrow{G}_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2}+\lambda\varepsilon_{\underline{\gamma}_1\underline{\alpha}_1}\varepsilon_{\underline{\gamma}_2\underline{\alpha}_2}=\kappa E_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2},$$

with $\overleftarrow{G}_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2}$ from (1.98), or, by using the d-tensor (1.99),

$$\Phi_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2}+(\frac{\overleftarrow{R}}{4}-\frac{\lambda}{2})\varepsilon_{\underline{\gamma}_1\underline{\alpha}_1}\varepsilon_{\underline{\gamma}_2\underline{\alpha}_2}=-\frac{\kappa}{2}E_{\underline{\gamma}_1\underline{\gamma}_2\underline{\alpha}_1\underline{\alpha}_2},$$

which are the d-spinor equivalent of the equations (1.44). These equations must be completed by the algebraic equations (1.45) for the d-torsion and d-spin density with d-tensor indices changed into corresponding d-spinor ones.

CHAPTER II

GAUGE FIELDS AND LOCALLY ANISOTROPIC GRAVITY

The aim of this Chapter is twofold. The first objective is to develop some of our results [Vacaru and Goncharenko 1995] on formulation of a geometrical approach to interactions of Yang-Mills fields on spaces with local anisotropy in the framework of the theory of linear connections in vector bundles (with semisimple structural groups) on la-spaces. The second objective is to extend the geometrical formalism in a manner including theories with nonsemisimple groups which permit a unique fiber bundle treatment for both locally anisotropic Yang-Mills field and gravitational interactions. In general lines, we shall follow the ideas and geometric methods proposed in [Bishop and Crittenden 1964], [Popov 1975], [Popov and Dikhin 1975], [Tseytlin 1982] and [Ponomariov, Barvinsky and Obukhov 1985], but we shall apply them in a form convenient for introducing into consideration geometrical constructions and physical theories on la-spaces.

There is a number of works on gauge models of interactions on Finsler spaces and theirs extensions(see, for instance, [Asanov 1985], [Bejancu 1990] and [Ono and Takano 1993]). One has introduced different variants of generalized gauge transforms, postulated corresponding Lagrangians for gravitational, gauge and matter field interactions and formulated variational calculus). The main problem of such models is the dependence of the basic equations on chosen definition of gauge "compensation" symmetries and on type of space and field interactions anisotropy. In order to avoid the ambiguities connected with particular characteristics of possible la-gauge theories we consider a "pure" geometric approach to gauge theories (on both locally isotropic and anisotropic spaces) in the framework of the theory of fiber bundles provided in general with different types of nonlinear and linear multiconnection and metric structures). This way, based on global geometric methods, holds also good for nonvariational, in the total spaces of bundles, gauge theories (in the case of gauge gravity based on Poincare or affine gauge groups); physical values and motion (field) equations have adequate geometric interpretation and do not depend on the type of local anisotropy of space-time background. It should be emphasized here that extensions for "higher order spaces" (see [Yano and Ishihara 1973] and [Miron and Atanasiu 1995]) can be realized in a straightforward manner.

II.1 Gauge Fields on Locally Anisotropic Spaces

This section is devoted to formulation of the geometrical background for gauge field theories on spaces with local anisotropy.

Let $(P, \pi, Gr, \mathcal{E})$ be a principal bundle on base $\mathcal{E}$ (being a la–space) with structural group Gr and surjective map $\pi : P \to \mathcal{E}$. At every point $u = (x, y) \in \mathcal{E}$ there is a vicinity $\mathcal{U} \subset E, u \in \mathcal{U}$, with trivializing P diffeomorphisms f and φ :

$$f_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times Gr, \qquad f(p) = (\pi(p), \varphi(p)),$$

$$\varphi_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \to Gr, \varphi(pq) = \varphi(p)\, q, \quad \forall q \in Gr,\ p \in P.$$

We remark that in the general case for two open regions

$$\mathcal{U}, \mathcal{V} \subset \mathcal{E}, \mathcal{U} \cap \mathcal{V} \neq \emptyset, f_{\mathcal{U}|_p} \neq f_{\mathcal{V}|_p}, \text{ even } p \in \mathcal{U} \cap \mathcal{V}.$$

Transition functions $g_{\mathcal{U}\mathcal{V}}$ are defined as

$$g_{\mathcal{U}\mathcal{V}} : \mathcal{U} \cap \mathcal{V} \to Gr, g_{\mathcal{U}\mathcal{V}}(u) = \varphi_{\mathcal{U}}(p) \left(\varphi_{\mathcal{V}}(p)^{-1}\right), \pi(p) = u.$$

Hereafter we shall omit, for simplicity, the specification of trivializing regions of maps and denote, for example, $f \equiv f_{\mathcal{U}}, \varphi \equiv \varphi_{\mathcal{U}},\ s \equiv s_{\mathcal{U}}$, if this will not give rise to ambiguities.

Let θ be the canonical left invariant 1-form on Gr with values in algebra Lie $\mathcal{G}$ of group Gr uniquely defined from the relation $\theta(q) = q, \forall q \in \mathcal{G}$, and consider a 1-form ω on $\mathcal{U} \subset \mathcal{E}$ with values in $\mathcal{G}$. Using θ and ω, we can locally define the connection form Ω in P as a 1-form:

$$\Omega = \varphi^* \theta + Ad\ \varphi^{-1}(\pi^* \omega) \tag{2.1}$$

where $\varphi^*\theta$ and $\pi^*\omega$ are, respectively, forms induced on $\pi^{-1}(\mathcal{U})$ and P by maps φ and π and $\omega = s^*\Omega$. The adjoint action on a form λ with values in $\mathcal{G}$ is defined as

$$\left(Ad\ \varphi^{-1} \lambda\right)_p = \left(Ad\ \varphi^{-1}(p)\right) \lambda_p$$

where λ_p is the value of form λ at point $p \in P$.

Introducing a basis $\{\Delta_{\widehat{a}}\}$ in $\mathcal{G}$ (index $\widehat{a}$ enumerates the generators making up this basis), we write the 1-form ω on $\mathcal{E}$ as

$$\omega = \Delta_{\widehat{a}} \omega^{\widehat{a}}(u),\ \omega^{\widehat{a}}(u) = \omega^{\widehat{a}}_{\mu}(u)\, \delta u^{\mu} \tag{2.2}$$

where $\delta u^{\mu} = \left(dx^i, \delta y^a\right)$ and the Einstein summation rule on indices $\widehat{a}$ and μ is used. Functions $\omega^{\widehat{a}}_{\mu}(u) = \omega^{\widehat{a}}_{\mu}(x, y)$ from (2.2) will be called the components of Yang-Mills fields on la-space $\mathcal{E}$. Gauge transforms of ω can be geometrically interpreted as transition relations for $\omega_{\mathcal{U}}$ and $\omega_{\mathcal{V}}$, when $u \in \mathcal{U} \cap \mathcal{V}$,

$$(\omega_{\mathcal{U}})_u = (g^*_{\mathcal{U}\mathcal{V}} \theta)_u + Ad\ g_{\mathcal{U}\mathcal{V}}(u)^{-1} (\omega_{\mathcal{V}})_u\,. \tag{2.3}$$

To relate $\omega^{\widehat{a}}_{\mu}$ with a covariant derivation we shall consider a vector bundle Υ associated to P. Let $\rho : Gr \to GL(\mathbb{R}^m)$ and $\rho' : \mathcal{G} \to End(E^m)$ be, respectively, linear representations of group Gr and Lie algebra $\mathcal{G}$ (in a more general

case we can consider $\mathbb{C}^m$ instead of $\mathbb{R}^m$). Map ρ defines a left action on Gr and associated vector bundle $\Upsilon = P \times \mathbb{R}^m / Gr,\ \pi_E : E \to \mathcal{E}$. Introducing the standard basis $\xi_{\underline{i}} = \{\xi_{\underline{1}}, \xi_{\underline{2}}, ..., \xi_{\underline{m}}\}$ in $\mathbb{R}^m$, we can define the right action on $P \times \mathbb{R}^m$, $\left((p,\xi)\, q = \left(pq, \rho\left(q^{-1}\right)\xi\right), q \in Gr\right)$, the map induced from P

$$p : \mathbb{R}^m \to \pi_E^{-1}(u), \quad (p(\xi) = (p\xi)\, Gr, \xi \in \mathbb{R}^m, \pi(p) = u)$$

and a basis of local sections $e_{\underline{i}} : U \to \pi_E^{-1}(U),\ e_{\underline{i}}(u) = s(u)\,\xi_{\underline{i}}$. Every section $\varsigma : \mathcal{E} \to \Upsilon$ can be written locally as $\varsigma = \varsigma^i e_i, \varsigma^i \in C^\infty(\mathcal{U})$. To every vector field X on $\mathcal{E}$ and Yang-Mills field $\omega^{\widehat{a}}$ on P we associate operators of covariant derivations:

$$\nabla_X \zeta = e_{\underline{i}}\left[X\zeta^{\underline{i}} + B(X)^{\underline{i}}_{\underline{j}}\,\zeta^{\underline{j}}\right] \text{ and } B(X) = (\rho' X)_{\widehat{a}}\,\omega^{\widehat{a}}(X). \tag{2.4}$$

Transformation laws (2.3) and operators (2.4) are interrelated by these transition transforms for values $e_{\underline{i}}, \zeta^{\underline{i}}$, and B_μ :

(2.5)
$$e^{\mathcal{V}}_{\underline{i}}(u) = \left[\rho g_{\mathcal{U}\mathcal{V}}(u)\right]^{\underline{j}}_{\underline{i}}\, e^{\mathcal{U}}_{\underline{i}},\ \zeta^{\underline{i}}_{\mathcal{U}}(u) = \left[\rho g_{\mathcal{U}\mathcal{V}}(u)\right]^{\underline{j}}_{\underline{i}}\,\zeta^{\underline{i}}_{\mathcal{V}},$$
$$B^{\mathcal{V}}_\mu(u) = \left[\rho g_{\mathcal{U}\mathcal{V}}(u)\right]^{-1}\delta_\mu\left[\rho g_{\mathcal{U}\mathcal{V}}(u)\right] + \left[\rho g_{\mathcal{U}\mathcal{V}}(u)\right]^{-1} B^{\mathcal{U}}_\mu(u)\left[\rho g_{\mathcal{U}\mathcal{V}}(u)\right],$$

where $B^{\mathcal{U}}_\mu(u) = B^\mu\left(\delta/du^\mu\right)(u)$.

Using (2.5), we can verify that the operator $\nabla^{\mathcal{U}}_X$, acting on sections of $\pi_\Upsilon : \Upsilon \to \mathcal{E}$ according to definition (2.4), satisfies the properties

$$\nabla^{\mathcal{U}}_{f_1X+f_2Y} = f_1\nabla^{\mathcal{U}}_X + f_2\nabla^{\mathcal{U}}_X,\ \nabla^{\mathcal{U}}_X(f\zeta) = f\nabla^{\mathcal{U}}_X\zeta + (Xf)\,\zeta,$$
$$\nabla^{\mathcal{U}}_X\zeta = \nabla^{\mathcal{V}}_X\zeta, \quad u \in \mathcal{U} \cap \mathcal{V}, f_1, f_2 \in C^\infty(\mathcal{U}).$$

So, we can conclude that the Yang_Mills connection in the vector bundle $\pi_\Upsilon : \Upsilon \to \mathcal{E}$ is not a general one, but is induced from the principal bundle $\pi : P \to \mathcal{E}$ with structural group Gr.

The curvature $\mathcal{K}$ of connection Ω from (2.1) is defined as

$$\mathcal{K} = D\Omega,\ D = \widehat{H} \circ d \tag{2.6}$$

where d is the operator of exterior derivation acting on $\mathcal{G}$–valued forms as

$$d\left(\Delta_{\widehat{a}} \otimes \chi^{\widehat{a}}\right) = \Delta_{\widehat{a}} \otimes d\chi^{\widehat{a}}$$

and $\widehat{H}$ is the horizontal projecting operator actin, for example, on the 1-form λ as $\left(\widehat{H}\lambda\right)_P(X_p) = \lambda_p(H_pX_p)$, where H_p projects on the horizontal subspace $\mathcal{H}_p \in P_p\left[X_p \in \mathcal{H}_p\right.$ is equivalent to $\left.\Omega_p(X_p) = 0\right]$. We can express (2.6) locally as

$$\mathcal{K} = Ad\ \varphi^{-1}_{\mathcal{U}}\left(\pi^*\mathcal{K}_{\mathcal{U}}\right) \tag{2.7}$$

where

$$\mathcal{K}_{\mathcal{U}} = d\omega_{\mathcal{U}} + \frac{1}{2}\left[\omega_{\mathcal{U}}, \omega_{\mathcal{U}}\right]. \tag{2.8}$$

The exterior product of $\mathcal{G}$–valued form (2.8) is defined as

$$\left[\Delta_{\widehat{a}} \otimes \lambda^{\widehat{a}}, \Delta_{\widehat{b}} \otimes \xi^{\widehat{b}}\right] = \left[\Delta_{\widehat{a}}, \Delta_{\widehat{b}}\right] \otimes \lambda^{\widehat{a}} \bigwedge \xi^{\widehat{b}},$$

where $\lambda^{\widehat{a}} \bigwedge \xi^{\widehat{b}} = \lambda^{\widehat{a}}\xi^{\widehat{b}} - \xi^{\widehat{b}}\lambda^{\widehat{a}}$. is the antisymmetric tensor product.

Introducing structural coefficients $f_{\widehat{bc}}{}^{\widehat{a}}$ of $\mathcal{G}$ satisfying relations $\left[\Delta_{\widehat{b}}, \Delta_{\widehat{c}}\right] = f_{\widehat{bc}}{}^{\widehat{a}}\Delta_{\widehat{a}}$ we can rewrite (2.8) in a form more convenient for local considerations:

$$\mathcal{K}_{\mathcal{U}} = \Delta_{\widehat{a}} \otimes \mathcal{K}^{\widehat{a}}_{\mu\nu}\delta u^{\mu} \bigwedge \delta u^{\nu} \tag{2.9}$$

where

$$\mathcal{K}^{\widehat{a}}_{\mu\nu} = \frac{\delta\omega^{\widehat{a}}_{\nu}}{\delta u^{\mu}} - \frac{\delta\omega^{\widehat{a}}_{\mu}}{\delta u^{\nu}} + \frac{1}{2} f_{\widehat{bc}}{}^{\widehat{a}}\left(\omega^{\widehat{b}}_{\mu}\omega^{\widehat{c}}_{\nu} - \omega^{\widehat{b}}_{\nu}\omega^{\widehat{c}}_{\mu}\right).$$

This section ends by considering the problem of reduction of the local anisotropic gauge symmetries and gauge fields to isotropic ones. For local trivial considerations we can consider that the vanishing of dependencies on y variables leads to isotropic Yang-Mills fields with the same gauge group as in the anisotropic case, Global geometric constructions require a more rigorous topological study of possible obstacles for reduction of total spaces and structural groups on anisotropic bases to their analogous on isotropic (for example, pseudo-Riemannian) base spaces.

II.2 Yang–Mills Equations on Locally Anisotropic Spaces

Interior gauge (nongravitational) symmetries are associated to semisimple structural groups. On the principal bundle $(P, \pi, Gr, \mathcal{E})$ with nondegenerate Killing form for semisimple group Gr we can define the generalized Lagrange metric

$$h_p\left(X_p, Y_p\right) = G_{\pi(p)}\left(d\pi_P X_P, d\pi_P Y_P\right) + K\left(\Omega_P\left(X_P\right), \Omega_P\left(X_P\right)\right), \tag{2.10}$$

where $d\pi_P$ is the differential of map $\pi : P \to \mathcal{E}$, $G_{\pi(p)}$ is locally generated as the la-metric (1.12), and K is the Killing form on $\mathcal{G}$:

$$K\left(\Delta_{\widehat{a}}, \Delta_{\widehat{b}}\right) = f_{\widehat{bd}}{}^{\widehat{c}} f_{\widehat{ac}}{}^{\widehat{d}} = K_{\widehat{ab}}.$$

Using the metric $G_{\alpha\beta}$ on $\mathcal{E}$ $\left[h_P\left(X_P, Y_P\right) \text{ on } P\right]$, we can introduce operators $*_G$ and $\widehat{\delta}_G$ acting in the space of forms on $\mathcal{E}$ ($*_H$ and $\widehat{\delta}_H$ acting on forms on $\mathcal{E}$). Let $e_{\underline{\mu}}$ be orthonormalized frames on $\mathcal{U} \subset \mathcal{E}$ and e^{μ} the adjoint coframes. Locally

$$G = \sum_{\mu} \eta\left(\mu\right) e^{\mu} \otimes e^{\mu},$$

where $\eta_{\mu\mu} = \eta(\mu) = \pm 1$, $\mu = 1, 2, ..., n, n+1, ..., n+m$, and the Hodge operator $*_G$ can be defined as $*_G : \Lambda'(\mathcal{E}) \to \Lambda^{n+m}(\mathcal{E})$, or, in explicit form, as

$$*_G\left(e^{\mu_1}\bigwedge ...\bigwedge e^{\mu_r}\right) = \eta(\nu_1)...\eta(\nu_{n+m-r}) \times sign\begin{pmatrix} 1 & 2 & ...r & r+1 & ...n+m \\ \mu_1 & \mu_2 & ...\mu_r & \nu_1 & ...\nu_{n+m-r}\end{pmatrix} \times e^{\nu_1}\bigwedge ...\bigwedge e^{\nu_{n+m-r}}. \tag{2.11}$$

Next, define the operator

$$*_G^{-1} = \eta(1)...\eta(n+m)(-1)^{r(n+m-r)} *_G$$

and introduce the scalar product on forms $\beta_1, \beta_2, ... \subset \Lambda^r(\mathcal{E})$ with compact carrier:

$$(\beta_1, \beta_2) = \eta(1)...\eta(n+m)\int \beta_1 \bigwedge *_G\beta_2.$$

The operator $\widehat{\delta}_G$ is defined as the adjoint to d associated to the scalar product for forms, specified for r-forms as

$$\widehat{\delta}_G = (-1)^r *_G^{-1} \circ d \circ *_G. \tag{2.12}$$

We remark that operators $*_H$ and δ_H acting in the total space of P can be defined similarly to (2.11) and (2.12), but by using metric (2.10). Both these operators also act in the space of $\mathcal{G}$-valued forms:

$$*\left(\Delta_{\widehat{a}} \otimes \varphi^{\widehat{a}}\right) = \Delta_{\widehat{a}} \otimes (*\varphi^{\widehat{a}}),\ \widehat{\delta}\left(\Delta_{\widehat{a}} \otimes \varphi^{\widehat{a}}\right) = \Delta_{\widehat{a}} \otimes (\widehat{\delta}\varphi^{\widehat{a}}).$$

The form λ on P with values in $\mathcal{G}$ is called horizontal if $\widehat{H}\lambda = \lambda$ and equivariant if $R^*(q)\lambda = Ad\ q^{-1}\varphi,\ \forall g \in Gr, R(q)$ being the right shift on P. We can verify that equivariant and horizontal forms also satisfy the conditions

$$\lambda = Ad\ \varphi_{\mathcal{U}}^{-1}(\pi^*\lambda), \qquad \lambda_{\mathcal{U}} = S_{\mathcal{U}}^*\lambda,\ (\lambda_{\mathcal{V}})_{\mathcal{U}} = Ad\ (g_{\mathcal{UV}}(u))^{-1}(\lambda_{\mathcal{U}})_u.$$

Definition 2.1. *The field equations for curvature (2.7) and connection (2.1) are defined by using geometric operators (2.11) and (2.12):*

$$\Delta\mathcal{K} = 0, \tag{2.13}$$

$$\nabla\mathcal{K} = 0, \tag{2.14}$$

where $\Delta = \widehat{H} \circ \widehat{\delta}_H$.

Equations (2.13) are similar to the well-known Maxwell equations and for non-Abelian gauge fields are called Yang-Mills equations. The structural equations (2.14) are called Bianchi identities.

The field equations (2.13) do not have a physical meaning because they are written in the total space of bundle Υ and not on the base anisotropic space-time $\mathcal{E}$. But this difficulty may be obviated by projecting the mentioned equations on the base. The 1-form $\Delta\mathcal{K}$ is horizontal by definition and its equivariance follows from the right invariance of metric (2.10). So, there is a unique form $(\Delta\mathcal{K})_{\mathcal{U}}$ satisfying

$$\Delta\mathcal{K} = Ad\ \varphi_{\mathcal{U}}^{-1}\pi^{*}(\Delta\mathcal{K})_{\mathcal{U}}.$$

Projection of (2.13) on the base can be written as $(\Delta\mathcal{K})_{\mathcal{U}} = 0$. To calculate $(\Delta\mathcal{K})_{\mathcal{U}}$, we use the equality [Bishop and Crittenden 1964] and [Popov and Dikhin 1975]

$$d\left(Ad\ \varphi_{\mathcal{U}}^{-1}\lambda\right) = Ad\ \varphi_{\mathcal{U}}^{-1}\ d\lambda - \left[\varphi_{\mathcal{U}}^{*}\theta, Ad\ \varphi_{\mathcal{U}}^{-1}\lambda\right] \tag{2.15}$$

where λ is a form on P with values in $\mathcal{G}$. For r–forms we have

$$\widehat{\delta}\left(Ad\ \varphi_{\mathcal{U}}^{-1}\lambda\right) = Ad\ \varphi_{\mathcal{U}}^{-1}\widehat{\delta}\lambda - (-1)^{r} *_{H} \{\left[\varphi_{\mathcal{U}}^{*}\theta, *_{H} Ad\ \varphi_{\mathcal{U}}^{-1}\lambda\right]$$

and, as a consequence,

$$\widehat{\delta}\mathcal{K} = Ad\ \varphi_{\mathcal{U}}^{-1}\{\widehat{\delta}_{H}\pi^{*}\mathcal{K}_{\mathcal{U}} + *_{H}^{-1}[\pi^{*}\omega_{\mathcal{U}}, *_{H}\pi^{*}\mathcal{K}_{\mathcal{U}}]\} - *_{H}^{-1}\left[\Omega, Ad\ \varphi_{\mathcal{U}}^{-1} *_{H}(\pi^{*}\mathcal{K})\right]. \tag{2.16}$$

By using straightforward calculations in a locally adapted dual basis on $\pi^{-1}(\mathcal{U})$ we can verify the equalities

$$\left[\Omega, Ad\ \varphi_{\mathcal{U}}^{-1}\ *_{H}(\pi^{*}\mathcal{K}_{\mathcal{U}})\right] = 0, \widehat{H}\delta_{H}(\pi^{*}\mathcal{K}_{\mathcal{U}}) = \pi^{*}\left(\widehat{\delta}_{G}\mathcal{K}\right), \quad *_{H}^{-1}[\pi^{*}\omega_{\mathcal{U}}, *_{H}(\pi^{*}\mathcal{K}_{\mathcal{U}})] = \pi^{*}\{*_{G}^{-1}[\omega_{\mathcal{U}}, *_{G}\mathcal{K}_{\mathcal{U}}]\}. \tag{2.17}$$

From (2.16) and (2.17) it follows that

$$(\Delta\mathcal{K})_{\mathcal{U}} = \widehat{\delta}_{G}\mathcal{K}_{\mathcal{U}} + *_{G}^{-1}[\omega_{\mathcal{U}}, *_{G}\mathcal{K}_{\mathcal{U}}]. \tag{2.18}$$

Taking into account (2.18) and (2.12), we prove that projection on $\mathcal{E}$ of equations (2.13) and (2.14) can be expressed respectively as

$$*_{G}^{-1}\circ d\circ *_{G}\mathcal{K}_{\mathcal{U}} + *_{G}^{-1}[\omega_{\mathcal{U}}, *_{G}\mathcal{K}_{\mathcal{U}}] = 0. \tag{2.19}$$

$$d\mathcal{K}_{\mathcal{U}} + [\omega_{\mathcal{U}}, \mathcal{K}_{\mathcal{U}}] = 0. \tag{2.20}$$

Equations (2.19) (see (2.18)) are gauge-invariant because

$$(\Delta\mathcal{K})_{\mathcal{U}} = Ad\ g_{\mathcal{U}\mathcal{V}}^{-1}(\Delta\mathcal{K})_{\mathcal{V}}.$$

By using formulas (2.9)-(2.12) we can rewrite (2.19) in coordinate form

$$D_{\nu}\left(G^{\nu\lambda}\mathcal{K}^{\widehat{a}}{}_{\lambda\mu}\right) + f_{\widehat{b}\widehat{c}}{}^{\widehat{a}}G^{\nu\lambda}\omega_{\lambda}^{\widehat{b}}\mathcal{K}^{\widehat{c}}{}_{\nu\mu} = 0, \tag{2.21}$$

where D_ν is, for simplicity, a compatible with metric covariant derivation on la-space $\mathcal{E}$.

It is possible to distinguish the x and y parts of equations (2.21) by using formulas (1.4),(1.5),(1.12),(1.17),(1.18),(1.28)(1.32). We omit this trivial calculus.

We point out that for our bundles with semisimple structural groups the Yang-Mills equations (2.13) (and, as a consequence, their horizontal projections (2.19) or (2.21)) can be obtained by variation of the action

$$I = \int \mathcal{K}^{\widehat{a}}{}_{\mu\nu} \mathcal{K}^{\widehat{b}}{}_{\alpha\beta} G^{\mu\alpha} G^{\nu\beta} K_{\widehat{a}\widehat{b}} |G_{\alpha\beta}|^{1/2}\, dx^1...dx^n \delta y^1 ... \delta y^m. \tag{2.22}$$

Equations for extremals of (2.22) have the form

$$K_{\widehat{r}\widehat{b}} G^{\lambda\alpha} G^{\kappa\beta} D_\alpha \mathcal{K}^{\widehat{b}}{}_{\lambda\beta} - K_{\widehat{a}\widehat{b}} G^{\kappa\alpha} G^{\nu\beta} f_{\widehat{r}\widehat{l}}{}^{\widehat{a}} \omega^{\widehat{l}}_{\nu} \mathcal{K}^{\widehat{b}}{}_{\alpha\beta} = 0, \tag{2.23}$$

which are equivalent to "pure" geometric equations (2.21) (or (2.19)) due to non-degeneration of the Killing form $K_{\widehat{r}\widehat{b}}$ for semisimple groups.

To take into account gauge interactions with matter fields (sections of vector bundle Υ on $\mathcal{E}$) we have to introduce a source 1-form $\mathcal{J}$ in equations (2.13) and to write them as

$$\Delta \mathcal{K} = \mathcal{J} \tag{2.24}$$

Explicit constructions of $\mathcal{J}$ require concrete definitions of the bundle Υ; for example, for spinor fields an invariant formulation of the Dirac equations on la-spaces is necessary. We omit spinor considerations in this Chapter (see section 1.7), but we shall present the definition of the source $\mathcal{J}$ for gravitational interactions (by using the energy–momentum tensor of matter on la–space) in the next section.

II.3 Gauge Locally Anisotropic Gravity

A considerable body of work on the formulation of gauge gravitational models on isotropic spaces is based on using nonsemisimple groups, for example, Poincare and affine groups, as structural gauge groups (see critical analysis and original results in [Tseytlin 1982] and [Ponomarev, Barvinsky and Obukhov 1985]). The main impediment to developing such models is caused by the degeneration of Killing forms for nonsemisimple groups, which make it impossible to construct consistent variational gauge field theories (functional (2.22) and extremal equations are degenarate in these cases). There are at least two possibilities to get around the mentioned difficulty. The first is to realize a minimal extension of the nonsemisimple group to a semisimple one, similar to the extension of the Poincare group to the de Sitter group (in the next section we shall use this operation for the definition of locally anisotropic gravitational instantons). The second possibility is to introduce into consideration the bundle of adapted affine frames on la–space $\mathcal{E}$, to use an auxiliary nondegenerate bilinear form $a_{\widehat{a}\widehat{b}}$ instead of the degenerate Killing form $K_{\widehat{a}\widehat{b}}$ and to consider a "pure" geometric method, illustrated in the previous section, of defining

gauge field equations. Projecting on the base $\mathcal{E}$, we shall obtain gauge gravitational field equations on la–space having a form similar to Yang-Mills equations.

The goal of this section is to prove that a specific parametrization of components of the Cartan connection in the bundle of adapted affine frames on $\mathcal{E}$ establishes an equivalence between Yang-Mills equations (2.24) and Einstein equations on la–spaces.

II.3.1 The bundle of linear locally adapted frames .

Let $(X_\alpha)_u = (X_i, X_a)_u$ be an adapted frame (see (1.4) at point $u \in \mathcal{E}$. We consider a local right distinguished action of matrices

$$A_{\alpha'}{}^{\alpha} = \begin{pmatrix} A_{i'}{}^{i} & 0 \\ 0 & B_{a'}{}^{a} \end{pmatrix} \subset GL_{n+m} = GL\left(n, \mathbb{R}\right) \oplus GL\left(m, \mathbb{R}\right).0$$

Nondegenerate matrices $A_{i'}{}^{i}$ and $B_{j'}{}^{j}$ respectively transforms linearly $X_{i|u}$ into $X_{i'|u} = A_{i'}{}^{i} X_{i|u}$ and $X_{a'|u}$ into $X_{a'|u} = B_{a'}{}^{a} X_{a|u}$, where $X_{\alpha'|u} = A_{\alpha'}{}^{\alpha} X_\alpha$ is also an adapted frame at the same point $u \in \mathcal{E}$. We denote by $La\left(\mathcal{E}\right)$ the set of all adapted frames X_α at all points of $\mathcal{E}$ and consider the surjective map π from $La\left(\mathcal{E}\right)$ to $\mathcal{E}$ transforming every adapted frame $X_{\alpha|u}$ and point u into point u. Every $X_{\alpha'|u}$ has a unique representation as $X_{\alpha'} = A_{\alpha'}{}^{\alpha} X_\alpha^{(0)}$, where $X_\alpha^{(0)}$ is a fixed distinguished basis in tangent space $T\left(\mathcal{E}\right)$. It is obvious that $\pi^{-1}\left(\mathcal{U}\right), \mathcal{U} \subset \mathcal{E}$, is bijective to $\mathcal{U} \times GL_{n+m}\left(\mathbb{R}\right)$. We can transform $La\left(\mathcal{E}\right)$ in a differentiable manifold taking $\left(u^\beta, A_{\alpha'}{}^{\alpha}\right)$ as a local coordinate system on $\pi^{-1}\left(\mathcal{U}\right)$. Now, it is easy to verify that $\mathcal{L}a(\mathcal{E}) = (La(\mathcal{E}, \mathcal{E}, GL_{n+m}(\mathbb{R})))$ is a principal bundle. We call $\mathcal{L}a(\mathcal{E})$ the bundle of linear adapted frames on $\mathcal{E}$.

The next step is to identify the components of, for simplicity, compatible d-connection $\Gamma^\alpha_{\beta\gamma}$ on $\mathcal{E}$:

$$\Omega^{\widehat{a}}_{\mathcal{U}} = \omega^{\widehat{a}} = \{\omega^{\widehat{\alpha\beta}}{}_{\lambda} \doteq \Gamma^\alpha_{\beta\gamma}\}. \tag{2.25}$$

Introducing (2.25) in (2.18), we calculate the local 1-form

(2.26)

$$\left(\Delta\mathcal{R}^{(\Gamma)}\right)_{\mathcal{U}} = \Delta_{\widehat{\alpha}\widehat{\alpha_1}} \otimes \left(G^{\nu\lambda} D_\lambda \mathcal{R}^{\widehat{\alpha}\widehat{\alpha_1}}{}_{\nu\mu} + f^{\widehat{\alpha}\widehat{\alpha_1}}{}_{\widehat{\beta}\widehat{\beta_1}\widehat{\gamma}\widehat{\gamma_1}} G^{\nu\lambda} \omega^{\widehat{\beta}\widehat{\beta_1}}{}_{\lambda} \mathcal{R}^{\widehat{\gamma}\widehat{\gamma_1}}{}_{\nu\mu}\right) \delta u^\mu,$$

where

$$\Delta_{\widehat{\alpha}\widehat{\alpha_1}} = \begin{pmatrix} \Delta_{ii_1} & 0 \\ 0 & \Delta_{aa_1} \end{pmatrix}$$

is the standard distinguished basis in Lie algebra of matrices $\mathcal{G}l_{n+m}\left(\mathbb{R}\right)$ with $\left(\Delta_{ii_1}\right)_{jj_1} = \delta_{ij}\delta_{i_1 j_1}$ and $\left(\Delta_{aa_1}\right)_{bb_1}$ being respectively the standard bases in $\mathcal{G}l\left(\mathbb{R}^{n+m}\right)$. We have denoted the curvature of connection (2.25), considered in (2.26), as

$$\mathcal{R}^{(\Gamma)}_{\mathcal{U}} = \Delta_{\widehat{\alpha}\widehat{\alpha_1}} \otimes \mathcal{R}^{\widehat{\alpha}\widehat{\alpha_1}}{}_{\nu\mu} X^\nu \bigwedge X^\mu, \tag{2.27}$$

where $\mathcal{R}^{\widehat{\alpha}\widehat{\alpha_1}}{}_{\nu\mu} = R_{\alpha_1}{}^{\alpha}{}_{\nu\mu}$ (see curvatures (1.32)).

II.3.2 The bundle of affine locally adapted frames.

Besides $\mathcal{L}a(\mathcal{E})$ with la–space $\mathcal{E}$, another bundle is naturally related, the bundle of adapted affine frames with structural group $Af_{n+m}(\mathbb{R}) = GL_{n+m}(\mathcal{E}) \otimes \mathbb{R}^{n+m}$. Because as linear space the Lie Algebra $af_{n+m}(\mathbb{R})$ is a direct sum of $\mathcal{G}l_{n+m}(\mathbb{R})$ and $\mathbb{R}^{n+m}$, we can write forms on $\mathcal{A}a(\mathcal{E})$ as $\Theta = (\Theta_1, \Theta_2)$, where Θ_1 is the $\mathcal{G}l_{n+m}(\mathbb{R})$ component and Θ_2 is the $\mathbb{R}^{n+m}$ component of the form Θ. Connection (2.25), Ω in $\mathcal{L}a(\mathcal{E})$, induces the Cartan connection $\overline{\Omega}$ in $\mathcal{A}a(\mathcal{E})$; see the isotropic case in [Bishop and Crittenden 1964] and [Popov and Dikhin 1975]. This is the unique connection on $\mathcal{A}a(\mathcal{E})$ represented as $i^*\overline{\Omega} = (\Omega, \chi)$, where χ is the shifting form and $i : \mathcal{A}a \to \mathcal{L}a$ is the trivial reduction of bundles. If $s_{\mathcal{U}}^{(a)}$ is a local adapted frame in $\mathcal{L}a(\mathcal{E})$, then $\overline{s}_{\mathcal{U}}^{(0)} = i \circ s_{\mathcal{U}}$ is a local section in $\mathcal{A}a(\mathcal{E})$ and

$$(\overline{\Omega}_{\mathcal{U}}) = s_{\mathcal{U}}\Omega = (\Omega_{\mathcal{U}}, \chi_{\mathcal{U}}), \tag{2.28}$$

$$(\overline{\mathcal{R}}_{\mathcal{U}}) = s_{\mathcal{U}}\overline{\mathcal{R}} = \left(\mathcal{R}_{\mathcal{U}}^{(\Gamma)}, T_{\mathcal{U}}\right), \tag{2.29}$$

where $\chi = e_{\widehat{\alpha}} \otimes \chi^{\widehat{\alpha}}{}_{\mu} X^{\mu}, G_{\alpha\beta} = \chi^{\widehat{\alpha}}{}_{\alpha} \chi^{\widehat{\beta}}{}_{\beta} \eta_{\widehat{\alpha}\widehat{\beta}}$ ($\eta_{\widehat{\alpha}\widehat{\beta}}$ is diagonal with $\eta_{\widehat{\alpha}\widehat{\alpha}} = \pm 1$) is a frame decomposition of metric (1.12) on $\mathcal{E}, e_{\widehat{\alpha}}$ is the standard distinguished basis on $\mathbb{R}^{n+m}$, and the projection of torsion , $T_{\mathcal{U}}$, on base $\mathcal{E}$ is defined as

$$T_{\mathcal{U}} = d\chi_{\mathcal{U}} + \Omega_{\mathcal{U}} \bigwedge \chi_{\mathcal{U}} + \chi_{\mathcal{U}} \bigwedge \Omega_{\mathcal{U}} = e_{\widehat{\alpha}} \otimes \sum_{\mu<\nu} T^{\widehat{\alpha}}{}_{\mu\nu} X^{\mu} \bigwedge X^{\nu}. \tag{2.30}$$

For a fixed local adapted basis on $\mathcal{U} \subset \mathcal{E}$ we can identify components $T^{\widehat{a}}{}_{\mu\nu}$ of torsion (2.30) with components of torsion (1.28) on $\mathcal{E}$, i.e. $T^{\widehat{\alpha}}{}_{\mu\nu} = T^{\alpha}{}_{\mu\nu}$. By straightforward calculation we obtain

$$(\Delta\overline{\mathcal{R}})_{\mathcal{U}} = [(\Delta\mathcal{R}^{(\Gamma)})_{\mathcal{U}},\ (R\tau)_{\mathcal{U}} + (Ri)_{\mathcal{U}}], \tag{2.31}$$

where

$$(R\tau)_{\mathcal{U}} = \widehat{\delta}_G T_{\mathcal{U}} + *_G^{-1}[\Omega_{\mathcal{U}}, *_G T_{\mathcal{U}}], \quad (Ri)_{\mathcal{U}} = *_G^{-1}\left[\chi_{\mathcal{U}}, *_G \mathcal{R}_{\mathcal{U}}^{(\Gamma)}\right].$$

Form $(Ri)_{\mathcal{U}}$ from (2.31) is locally constructed by using components of the Ricci tensor (see (1.33)) as follows from decomposition on the local adapted basis $X^{\mu} = \delta u^{\mu}$:

$$(Ri)_{\mathcal{U}} = e_{\widehat{\alpha}} \otimes (-1)^{n+m+1} R_{\lambda\nu} G^{\widehat{\alpha}\lambda} \delta u^{\mu} \tag{2.32}$$

We remark that for isotropic torsionless pseudo-Riemannian spaces the requirement that $(\Delta\overline{\mathcal{R}})_{\mathcal{U}} = 0$, i.e., imposing the connection (2.25) to satisfy Yang-Mills equations (2.13) (equivalently (2.19) or (2.21) we obtain the equivalence of the mentioned gauge gravitational equations with the vacuum Einstein equations $R_{ij} = 0$. In the case of la–spaces with arbitrary given torsion, even considering vacuum gravitational fields, we have to introduce a source for gauge gravitational equations in order to compensate for the contribution of torsion and to obtain equivalence with the Einstein equations.

The above presented considerations constitute the proof of the following

Theorem II.1. *The Einstein equations (1.42) for locally anisotropic gravity are equivalent to Yang-Mills equations*

$$(\Delta\overline{\mathcal{R}}) = \overline{\mathcal{J}} \tag{2.33}$$

for the induced Cartan connection $\overline{\Omega}$ *(see (2.25), (2.28)) in the bundle of local adapted affine frames* $\mathcal{A}a(\mathcal{E})$ *with source* $\overline{\mathcal{J}}_{\mathcal{U}}$ *constructed locally by using the same formulas (2.31) (for* $(\Delta\overline{\mathcal{R}})$*), where* $R_{\alpha\beta}$ *is changed by the matter source* $\tilde{E}_{\alpha\beta} - \frac{1}{2}G_{\alpha\beta}\tilde{E}$*, where* $\tilde{E}_{\alpha\beta} = kE_{\alpha\beta} - \lambda G_{\alpha\beta}$.

II.4 Nonlinear De Sitter Gauge Locally Anisotropic Gravity

The equivalent reexpression of the Einstein theory as a gauge like theory implies, for both locally isotropic and anisotropic space-times, the nonsemisimplicity of the gauge group, which leads to a nonvariational theory in the total space of the bundle of locally adapted affine frames. A variational gauge gravitational theory can be formulated by using a minimal extension of the affine structural group $\mathcal{A}f_{n+m}(\mathbb{R})$ to the de Sitter gauge group $S_{n+m} = SO(n+m+1)$ acting on distinguished $\mathbb{R}^{n+m+1}$ space.

II.4.1 Nonlinear gauge theories of de Sitter group.

Let us consider the de Sitter space Σ^{n+m} as a hypersurface given by the equations $\eta_{AB}u^A u^B = -l^2$ in the (n+m)-dimensional spaces enabled with diagonal metric $\eta_{AB}, \eta_{AA} = \pm 1$ (in this section $A, B, C, ... = 1, 2, ..., n+m+1$), where $\{u^A\}$ are global Cartesian coordinates in $\mathbb{R}^{n+m+1}; l > 0$ is the curvature of de Sitter space. The de Sitter group $S_{(\eta)} = SO_{(\eta)}(n+m+1)$ is defined as the isometry group of Σ^{n+m}-space with $\frac{n+m}{2}(n+m+1)$ generators of Lie algebra $\int o_{(\eta)}(n+m+1)$ satisfying the commutation relations

$$[M_{AB}, M_{CD}] = \eta_{AC}M_{BD} - \eta_{BC}M_{AD} - \eta_{AD}M_{BC} + \eta_{BD}M_{AC}. \tag{2.34}$$

Decomposing indices $A, B, ...$ as $A = (\widehat{\alpha}, n+m+1), B = \left(\widehat{\beta}, n+m+1\right)$, ..., the metric η_{AB} as $\eta_{AB} = \left(\eta_{\widehat{\alpha}\widehat{\beta}}, \eta_{(n+m+1)(n+m+1)}\right)$, and operators M_{AB} as $M_{\widehat{\alpha}\widehat{\beta}} = \mathcal{F}_{\widehat{\alpha}\widehat{\beta}}$ and $P_{\widehat{\alpha}} = l^{-1}M_{n+m+1,\widehat{\alpha}}$, we can write (2.34) as

$$\left[\mathcal{F}_{\widehat{\alpha}\widehat{\beta}}, \mathcal{F}_{\widehat{\gamma}\widehat{\delta}}\right] = \eta_{\widehat{\alpha}\widehat{\gamma}}\mathcal{F}_{\widehat{\beta}\widehat{\delta}} - \eta_{\widehat{\beta}\widehat{\gamma}}\mathcal{F}_{\widehat{\alpha}\widehat{\delta}} + \eta_{\widehat{\beta}\widehat{\delta}}\mathcal{F}_{\widehat{\alpha}\widehat{\gamma}} - \eta_{\widehat{\alpha}\widehat{\delta}}\mathcal{F}_{\widehat{\beta}\widehat{\gamma}},$$

$$\left[P_{\widehat{\alpha}}, P_{\widehat{\beta}}\right] = -l^{-2}\mathcal{F}_{\widehat{\alpha}\widehat{\beta}}, \quad \left[P_{\widehat{\alpha}}, \mathcal{F}_{\widehat{\beta}\widehat{\gamma}}\right] = \eta_{\widehat{\alpha}\widehat{\beta}}P_{\widehat{\gamma}} - \eta_{\widehat{\alpha}\widehat{\gamma}}P_{\widehat{\beta}},$$

where we have indicated the possibility to decompose $\int o_{(\eta)}(n+m+1)$ into a direct sum, $\int o_{(\eta)}(n+m+1) = \int o_{(\eta)}(n+m) \oplus V_{n+m}$, where V_{n+m} is the vector space stretched on vectors $P_{\widehat{\alpha}}$. We remark that $\Sigma^{n+m} = S_{(\eta)}/L_{(\eta)}$, where $L_{(\eta)} =$

$SO_{(\eta)}\left(n+m\right)$. For $\eta_{AB}=diag\left(1,-1,-1,-1\right)$ and $S_{10}=SO\left(1,4\right), L_6=SO\left(1,3\right)$ is the group of Lorentz rotations.

Let $W\left(\mathcal{E},\mathbb{R}^{n+m+1},S_{(\eta)},P\right)$ be the vector bundle associated with principal bundle $P\left(S_{(\eta)},\mathcal{E}\right)$ on la–spaces. The action of the structural group $S_{(\eta)}$ on E can be realized by using $(n+m)\times(n+m)$ matrices with a parametrization distinguishing subgroup $L_{(\eta)}$:

$$B=bB_L,\quad \text{where } B_L=\begin{pmatrix} L & 0\\ 0 & 1\end{pmatrix}, \tag{2.35}$$

$L\in L_{(\eta)}$ is the de Sitter bust matrix transforming the vector $(0,0,...,\rho)\in\mathbb{R}^{n+m+1}$ into the arbitrary point $\left(V^1,V^2,...,V^{n+m+1}\right)\in\Sigma_\rho^{n+m}\subset\mathbb{R}^{n+m+1}$ with curvature $\rho\quad\left(V_AV^A=-\rho^2,V^A=t^A\rho\right)$. Matrix b can be expressed as

$$b=\begin{pmatrix} \delta^{\widehat{\alpha}}{}_{\widehat{\beta}}+\frac{t^{\widehat{\alpha}}t_{\widehat{\beta}}}{(1+t^{n+m+1})} & t^{\widehat{\alpha}}\\ t_{\widehat{\beta}} & t^{n+m+1}\end{pmatrix}.$$

The de Sitter gauge field is associated with a linear connection in W, i.e., with a $\int o_{(\eta)}\left(n+m+1\right)$-valued connection 1-form on $\mathcal{E}$:

$$\widetilde{\Omega}=\begin{pmatrix} \omega^{\widehat{\alpha}}{}_{\widehat{\beta}} & \widetilde{\theta}^{\widehat{\alpha}}\\ \widetilde{\theta}_{\widehat{\beta}} & 0\end{pmatrix}, \tag{2.36}$$

where $\omega^{\widehat{\alpha}}{}_{\widehat{\beta}}\in so(n+m)_{(\eta)}$, $\widetilde{\theta}^{\widehat{\alpha}}\in\mathbb{R}^{n+m}$, $\widetilde{\theta}_{\widehat{\beta}}\in\eta_{\widehat{\beta}\widehat{\alpha}}\widetilde{\theta}^{\widehat{\alpha}}$.

Because $S_{(\eta)}$-transforms mix $\omega^{\widehat{\alpha}}{}_{\widehat{\beta}}$ and $\widetilde{\theta}^{\widehat{\alpha}}$ fields in (2.36) (the introduced parametrization is invariant on action on $SO_{(\eta)}\left(n+m\right)$ group we cannot identify $\omega^{\widehat{\alpha}}{}_{\widehat{\beta}}$ and $\widetilde{\theta}^{\widehat{\alpha}}$, respectively, with the connection $\Gamma^\alpha{}_{\beta\gamma}$ and the fundamental form χ^α in $\mathcal{E}$ (as we have for (2.25) and (2.28)). To avoid this difficulty we consider [Tseytlin 1982] a nonlinear gauge realization of the de Sitter group $S_{(\eta)}$, namely, we introduce into consideration the nonlinear gauge field

$$\Omega=b^{-1}\Omega b+b^{-1}db=\begin{pmatrix} \Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}} & \theta^{\widehat{\alpha}}\\ \theta_{\widehat{\beta}} & 0\end{pmatrix}, \tag{2.37}$$

where

$$\Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}}=\omega^{\widehat{\alpha}}{}_{\widehat{\beta}}-\left(t^{\widehat{\alpha}}Dt_{\widehat{\beta}}-t_{\widehat{\beta}}Dt^{\widehat{\alpha}}\right)/\left(1+t^{n+m+1}\right),$$

$$\theta^{\widehat{\alpha}}=t^{n+m+1}\widetilde{\theta}^{\widehat{\alpha}}+Dt^{\widehat{\alpha}}-t^{\widehat{\alpha}}\left(dt^{n+m+1}+\widetilde{\theta}_{\widehat{\gamma}}t^{\widehat{\gamma}}\right)/\left(1+t^{n+m+1}\right),$$

$$Dt^{\widehat{\alpha}}=dt^{\widehat{\alpha}}+\omega^{\widehat{\alpha}}{}_{\widehat{\beta}}t^{\widehat{\beta}}.$$

The action of the group $S(\eta)$ is nonlinear, yielding transforms $\Gamma' = L'\Gamma(L')^{-1} + L'd(L')^{-1}, \theta' = L\theta$, where the nonlinear matrix-valued function $L' = L'(t^{\alpha}, b, B_T)$ is defined from $B_b = b'B_{L'}$ (see parametrization (2.35)).

Now, we can identify components of (2.37) with components of $\Gamma^{\alpha}{}_{\beta\gamma}$ and $\chi^{\widehat{\alpha}}{}_{\alpha}$ on $\mathcal{E}$ and induce in a consistent manner on the base of bundle $W\left(\mathcal{E}, \mathbb{R}^{n+m+1}, S_{(\eta)}, P\right)$ the la–geometry.

II.4.2 Dynamics of the nonlinear locally anisotropic De Sitter gravity. Instead of the gravitational potential (2.25), we introduce the gravitational connection (similar to (2.37))

$$\Gamma = \begin{pmatrix} \Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}} & l_0^{-1}\chi^{\widehat{\alpha}} \\ l_0^{-1}\chi_{\widehat{\beta}} & 0 \end{pmatrix} \tag{2.38}$$

where

$$\Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}} = \Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}\mu}\delta u^{\mu}, \Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}\mu} = \chi^{\widehat{\alpha}}{}_{\alpha}\chi^{\widehat{\beta}}{}_{\beta}\Gamma^{\alpha}{}_{\beta\gamma} + \chi^{\widehat{\alpha}}{}_{\alpha}\delta_{\mu}\chi^{\alpha}{}_{\widehat{\beta}}, \chi^{\widehat{\alpha}} = \chi^{\widehat{\alpha}}{}_{\mu}\delta u^{\mu},$$

$G_{\alpha\beta} = \chi^{\widehat{\alpha}}{}_{\alpha}\chi^{\widehat{\beta}}{}_{\beta}\eta_{\widehat{\alpha}\widehat{\beta}}$, $\eta_{\widehat{\alpha}\widehat{\beta}}$ is parametrized as $\eta_{\widehat{\alpha}\widehat{\beta}} = \begin{pmatrix} \eta_{ij} & 0 \\ 0 & \eta_{ab} \end{pmatrix}, \eta_{ij} = (1, -1, ..., -1)$, and l_0 is a dimensional constant.

The curvature of (2.39), $\mathcal{R}^{(\Gamma)} = d\Gamma + \Gamma \bigwedge \Gamma$, can be written as

$$\mathcal{R}^{(\Gamma)} = \begin{pmatrix} \mathcal{R}^{\widehat{\alpha}}{}_{\widehat{\beta}} + l_0^{-1}\pi^{\widehat{\alpha}}_{\widehat{\beta}} & l_0^{-1}T^{\widehat{\alpha}} \\ l_0^{-1}T^{\widehat{\beta}} & 0 \end{pmatrix}, \tag{2.39}$$

where

$$\pi^{\widehat{\alpha}}_{\widehat{\beta}} = \chi^{\widehat{\alpha}} \bigwedge \chi_{\widehat{\beta}}, \mathcal{R}^{\widehat{\alpha}}{}_{\widehat{\beta}} = \frac{1}{2}\mathcal{R}^{\widehat{\alpha}}{}_{\widehat{\beta}\mu\nu}\delta u^{\mu} \bigwedge \delta u^{\nu},$$

and

$$\mathcal{R}^{\widehat{\alpha}}{}_{\widehat{\beta}\mu\nu} = \chi_{\widehat{\beta}}{}^{\beta}\chi_{\alpha}{}^{\widehat{\alpha}}R^{\alpha}{}_{\beta_{\mu\nu}}$$

(see (1.31) and (1.32), the components of d-curvatures). The de Sitter gauge group is semisimple and we are able to construct a variational gauge gravitational locally anisotropic theory (bundle metric (2.10) is nondegenerate). The Lagrangian of the theory is postulated as

$$L = L_{(G)} + L_{(m)}$$

where the gauge gravitational Lagrangian is defined as

$$L_{(G)} = \frac{1}{4\pi}Tr\left(\mathcal{R}^{(\Gamma)} \bigwedge *_G\mathcal{R}^{(\Gamma)}\right) = \mathcal{L}_{(G)}\left|G\right|^{1/2}\delta^{n+m}u,$$

$$\mathcal{L}_{(G)} = \frac{1}{2l^2}T^{\widehat{\alpha}}{}_{\mu\nu}T_{\widehat{\alpha}}{}^{\mu\nu} + \frac{1}{8\lambda}\mathcal{R}^{\widehat{\alpha}}{}_{\widehat{\beta}\mu\nu}\mathcal{R}^{\widehat{\beta}}{}_{\widehat{\alpha}}{}^{\mu\nu} - \frac{1}{l^2}\left(\overleftarrow{R}(\Gamma) - 2\lambda_1\right), \tag{2.40}$$

$T^{\widehat{\alpha}}{}_{\mu\nu} = \chi^{\widehat{\alpha}}{}_{\alpha} T^{\alpha}{}_{\mu\nu}$ (the gravitational constant l^2 in (2.40) satisfies the relations $l^2 = 2l_0^2\lambda, \lambda_1 = -3/l_0]$), Tr denotes the trace on $\widehat{\alpha}, \widehat{\beta}$ indices, and the matter field Lagrangian is defined as

$$L_{(m)} = -1\frac{1}{2}Tr\left(\Gamma \bigwedge *_G \mathcal{I}\right) = \mathcal{L}_{(m)} \left|G\right|^{1/2} \delta^{n+m} u,$$

$$\mathcal{L}_{(m)} = \frac{1}{2}\Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}\mu} S^{\widehat{\beta}}{}_{\alpha}{}^{\mu} - t^{\mu}{}_{\widehat{\alpha}} l^{\widehat{\alpha}}{}_{\mu}. \tag{2.41}$$

The matter field source $\mathcal{I}$ is obtained as a variational derivation of $\mathcal{L}_{(m)}$ on Γ and is parametrized as

$$\mathcal{I} = \begin{pmatrix} S^{\widehat{\alpha}}{}_{\widehat{\beta}} & -l_0 t^{\widehat{\alpha}} \\ -l_0 t_{\widehat{\beta}} & 0 \end{pmatrix} \tag{2.42}$$

with $t^{\widehat{\alpha}} = t^{\widehat{\alpha}}{}_{\mu}\delta u^{\mu}$ and $S^{\widehat{\alpha}}{}_{\widehat{\beta}} = S^{\widehat{\alpha}}{}_{\widehat{\beta}\mu}\delta u^{\mu}$ being respectively the canonical tensors of energy-momentum and spin density. Because of the contraction of the "interior" indices $\widehat{\alpha}, \widehat{\beta}$ in (2.40) and (2.41) we used the Hodge operator $*_G$ instead of $*_H$ (hereafter we consider $*_G = *$).

Varying (by taking into account the distinguishing by N–connection) the action

$$S = \int \left|G\right|^{1/2} \delta^{n+m} u \left(\mathcal{L}_{(G)} + \mathcal{L}_{(m)}\right)$$

on the Γ-variables (2.31), we obtain the gauge-gravitational field equations:

$$d\left(*\mathcal{R}^{(\Gamma)}\right) + \Gamma \bigwedge \left(*\mathcal{R}^{(\Gamma)}\right) - \left(*\mathcal{R}^{(\Gamma)}\right) \bigwedge \Gamma = -\lambda \left(*\mathcal{I}\right). \tag{2.43}$$

Specifying the variations on $\Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}}$ and $l^{\widehat{\alpha}}$-variables, we rewrite (2.43) as

$$\widehat{\mathcal{D}}\left(*\mathcal{R}^{(\Gamma)}\right) + \frac{2\lambda}{l^2}\left(\widehat{\mathcal{D}}\left(*\pi\right) + \chi \bigwedge \left(*T^T\right) - \left(*T\right) \bigwedge \chi^T\right) = -\lambda \left(*S\right), \tag{2.44}$$

$$\widehat{\mathcal{D}}\left(*T\right) - \left(*\mathcal{R}^{(\Gamma)}\right) \bigwedge \chi - \frac{2\lambda}{l^2}\left(*\pi\right) \bigwedge \chi = \frac{l^2}{2}\left(*t + \frac{1}{\lambda} * \tau\right), \tag{2.45}$$

where

$$T^t = \{T_{\widehat{\alpha}} = \eta_{\widehat{\alpha}\widehat{\beta}} T^{\widehat{\beta}},\ T^{\widehat{\beta}} = \frac{1}{2} T^{\widehat{\beta}}{}_{\mu\nu} \delta u^{\mu} \bigwedge \delta u^{\nu}\},$$

$$\chi^T = \{\chi_{\widehat{\alpha}} = \eta_{\widehat{\alpha}\widehat{\beta}} \chi^{\widehat{\beta}},\ \chi^{\widehat{\beta}} = \chi^{\widehat{\beta}}{}_{\mu} \delta u^{\mu}\}, \qquad \widehat{\mathcal{D}} = d + \widehat{\Gamma}$$

($\widehat{\Gamma}$ acts as $\Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}\mu}$ on indices $\widehat{\gamma},\widehat{\delta},...$ and as $\Gamma^{\alpha}{}_{\beta\mu}$ on indices $\gamma,\delta,...$). In (2.45), τ defines the energy-momentum tensor of the $S_{(\eta)}$-gauge gravitational field $\widehat{\Gamma}$:

$$\tau_{\mu\nu}\left(\widehat{\Gamma}\right)=\frac{1}{2}Tr\left(\mathcal{R}_{\mu\alpha}\mathcal{R}^{\alpha}{}_{\nu}-\frac{1}{4}\mathcal{R}_{\alpha\beta}\mathcal{R}^{\alpha\beta}G_{\mu\nu}\right). \tag{2.46}$$

Equations (2.43) (or equivalently (2.44),(2.45)) make up the complete system of variational field equations for nonlinear de Sitter gauge gravity with local anisotropy. They can be interpreted as a generalization of Miron and Anastasiei equations for la–gravity [Vacaru and Goncharenko 1995] (equivalently, of gauge gravitational equations (2.33)] to a system of gauge field equations with dynamical torsion and corresponding spin–density source.

Finally, we remark that we can obtain a nonvariational Poincare gauge gravitational theory on la-spaces if we consider the contraction of the gauge potential (2.38) to a potential with values in the Poincare Lie algebra

$$\Gamma=\begin{pmatrix}\Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}} & l_0^{-1}\chi^{\widehat{\alpha}}\\ l_0^{-1}\chi_{\widehat{\beta}} & 0\end{pmatrix}\rightarrow\Gamma=\begin{pmatrix}\Gamma^{\widehat{\alpha}}{}_{\widehat{\beta}} & l_0^{-1}\chi^{\widehat{\alpha}}\\ 0 & 0\end{pmatrix}.$$

Isotropic Poincare gauge gravitational theories are studied in a number of papers (see, for example, references from [Tseytlin 1982] and [Ponomarev, Barvinsky and Obukhov 1985]). In a manner similar to considerations presented in this work, we can generalize Poincare gauge models for spaces with local anisotropy.

CHAPTER III

NEARLY AUTOPARALLEL MAPS AND CONSERVATION LAWS

The study of models of classical and quantum field interactions on la–spaces is in order of the day. The development of this branch of theoretical and mathematical physics entails great difficulties because of problematical character of the possibility and manner of definition of conservation laws on la–spaces. It will be recalled that, for instance, conservation laws of energy–momentum type are a consequence of existence of a global group of automorphisms of the fundamental Mikowski spaces (the tangent space's automorphisms and particular cases when there are symmetries generated by existence of Killing vectors are considered for (pseudo)Riemannian spaces). No global or local automorphisms exist on generic la-spaces and in result of this fact the formulation of la-conservation laws is sophisticate and full of ambiguities. R. Miron and M. Anastasiei firstly pointed out the nonzero divergence of the matter energy-momentum d–tensor, the source in Einstein equations on la–spaces, and considered an original approach to the geometry of time–dependent Lagrangians [Miron and Anastasiei 1994]. Nevertheless, the rigorous definition of energy-momentum values for la–gravitational and matter fields and the form of conservation laws for such values have not been considered in present–day studies of the mentioned problem.

The question of definition of tensor integration as the inverse operation of covariant derivation was posed and studied by [Moór 1951]. Tensor–integral and bitensor formalisms turned out to be very useful in solving certain problems connected with conservation laws in general relativity. In order to extend tensor–integral constructions we have proposed [Gottlieb and Vacaru 1996] to take into consideration nearly autoparallel [Vacaru 1992] and nearly geodesic [Sinyukov 1992] maps, ng–maps, which forms a subclass of local 1–1 maps of curved spaces with deformation of the connection and metric structures.

One of the main purposes of this Chapter is to synthesize the results on nearly autoparallel maps and tensor integral and to formulate them for a very general class of la–spaces. As the next step the la–gravity and an analysis of la–conservation laws are considered.

We note that proofs of our theorems are mechanical, but, in most cases, they are rather tedious calculations similar to those presented in [Sinyukov 1979], [Vacaru and Ostaf 1996a, 1996b] and [Gottlieb and Vacaru 1996]. Some of them, on la–spaces, will be given in detail the rest, being similar, or consequences, will be only sketched or omitted.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

III.1 Nearly Autoparallel Maps of Locally Anisotropic Spaces

In our geometric constructions we shall use pairs of open regions $(U, \underline{U})$ of la–spaces, $U \subset \xi$, $\underline{U} \subset \underline{\xi}$, and 1–1 local maps $f : U \rightarrow \underline{U}$ given by functions $f^a(u)$ of smoothly class $C^r(U)\,(r > 2$, or $r = \omega$ for analytic functions) and their inverse functions $f^{\underline{a}}(\underline{u})$ with corresponding non–zero Jacobians in every point $u \in U$ and $\underline{u} \in \underline{U}$.

We consider that two open regions U and $\underline{U}$ are attributed to a common for f–map coordinate system if this map is realized on the principle of coordinate equality $q(u^\alpha) \rightarrow \underline{q}(u^\alpha)$ for every point $q \in U$ and its f–image $\underline{q} \in \underline{U}$ and note that all calculations included in this Chapter will be local in nature and taken to refer to open subsets of mappings of type $\xi \supset U \xrightarrow{f} \underline{U} \subset \underline{\xi}$. For simplicity, we suppose that in a fixed common coordinate system for U and $\underline{U}$ spaces ξ and $\underline{\xi}$ are characterized by a common N–connection structure (in consequence of (1.10) by a corresponding concordance of d–metric structure), i.e.

$$N_j^a(u) = \underline{N}_j^a(u) = \underline{N}_j^a(\underline{u}),$$

which leads to the possibility to establish common local bases, adapted to a given N–connection, on both regions U and $\underline{U}$. We consider that on ξ it is defined the linear d–connection structure with components $\Gamma^\alpha_{.\beta\gamma}$. On the space $\underline{\xi}$ the linear d–connection is considered to be a general one with torsion

$$\underline{T}^\alpha_{.\beta\gamma} = \underline{\Gamma}^\alpha_{.\beta\gamma} - \underline{\Gamma}^\alpha_{.\gamma\beta} + w^\alpha_{.\beta\gamma}$$

and nonmetricity

$$\underline{K}_{\alpha\beta\gamma} = \underline{D}_\alpha \underline{G}_{\beta\gamma}. \tag{3.1}$$

Geometrical objects on $\underline{\xi}$ are specified by underlined symbols (for example, $\underline{A}^\alpha, \underline{B}^{\alpha\beta}$) or underlined indices (for example, $A^{\underline{a}}, B^{\underline{ab}}$).

For our purposes it is convenient to introduce auxiliary symmetric d–connections, $\gamma^\alpha_{.\beta\gamma} = \gamma^\alpha_{.\gamma\beta}$ on ξ and $\underline{\gamma}^\alpha_{.\beta\gamma} = \underline{\gamma}^\alpha_{.\gamma\beta}$ on $\underline{\xi}$ defined, correspondingly, as

$$\Gamma^\alpha_{.\beta\gamma} = \gamma^\alpha_{.\beta\gamma} + T^\alpha_{.\beta\gamma} \quad and \quad \underline{\Gamma}^\alpha_{.\beta\gamma} = \underline{\gamma}^\alpha_{.\beta\gamma} + \underline{T}^\alpha_{.\beta\gamma}.$$

We are interested in definition of local 1–1 maps from U to $\underline{U}$ characterized by symmetric, $P^\alpha_{.\beta\gamma}$, and antisymmetric, $Q^\alpha_{.\beta\gamma}$, deformations:

$$\underline{\gamma}^\alpha_{.\beta\gamma} = \gamma^\alpha_{.\beta\gamma} + P^\alpha_{.\beta\gamma} \tag{3.2}$$

and

$$\underline{T}^\alpha_{.\beta\gamma} = T^\alpha_{.\beta\gamma} + Q^\alpha_{.\beta\gamma}. \tag{3.3}$$

The auxiliary linear covariant derivations induced by $\gamma^\alpha_{.\beta\gamma}$ and $\underline{\gamma}^\alpha_{.\beta\gamma}$ are denoted respectively as ${}^{(\gamma)}D$ and ${}^{(\gamma)}\underline{D}$.

Let introduce this local coordinate parametrization of curves on U :

$$u^\alpha = u^\alpha(\eta) = (x^i(\eta), y^i(\eta)),\ \eta_1 < \eta < \eta_2,$$

where corresponding tangent vector field is defined as

$$v^\alpha = \frac{du^\alpha}{d\eta} = (\frac{dx^i(\eta)}{d\eta}, \frac{dy^j(\eta)}{d\eta}).$$

Definition 3.1. *A curve l is called auto parallel, a–parallel, on ξ if its tangent vector field v^α satisfies a–parallel equations:*

$$vDv^\alpha = v^{\beta(\gamma)} D_\beta v^\alpha = \rho(\eta) v^\alpha, \tag{3.4}$$

where $\rho(\eta)$ is a scalar function on ξ.

Let curve $\underline{l} \subset \underline{\xi}$ is given in parametric form as $u^\alpha = u^\alpha(\eta),\ \eta_1 < \eta < \eta_2$ with tangent vector field $v^\alpha = \frac{du^\alpha}{d\eta} \neq 0$. We suppose that a 2–dimensional distribution $E_2(\underline{l})$ is defined along $\underline{l}$, i.e. in every point $u \in \underline{l}$ is fixed a 2-dimensional vector space $E_2(\underline{l}) \subset \underline{\xi}$. The introduced distribution $E_2(\underline{l})$ is coplanar along $\underline{l}$ if every vector $\underline{p}^\alpha(u^b_{(0)}) \subset E_2(\underline{l}), u^\beta_{(0)} \subset \underline{l}$ rests contained in the same distribution after parallel transports along $\underline{l}$, i.e. $\underline{p}^\alpha(u^\beta(\eta)) \subset E_2(\underline{l})$.

Definition 3.2. *A curve $\underline{l}$ is called nearly autoparallel, or in brief an na–parallel, on space $\underline{\xi}$ if a coplanar along $\underline{l}$ distribution $E_2(\underline{l})$ containing tangent to $\underline{l}$ vector field $v^\alpha(\eta)$, i.e. $v^\alpha(\eta) \subset E_2(\underline{l})$, is defined.*

The nearly autoparallel maps of la–spaces are introduced according the definition:

Definition 3.3. *Nearly autoparallel maps, na–maps, of la–spaces are defined as local 1–1 mappings of v–bundles, $\xi \to \underline{\xi}$, changing every a–parallel on ξ into a na–parallel on $\underline{\xi}$.*

Now we formulate the general conditions when deformations (3.2) and (3.3) characterize na-maps : Let a-parallel $l \subset U$ is given by functions $u^\alpha = u^{(\alpha)}(\eta), v^\alpha = \frac{du^\alpha}{d\eta}$, $\eta_1 < \eta < \eta_2$, satisfying equations (3.4). We suppose that to this a–parallel corresponds a na–parallel $\underline{l} \subset \underline{U}$ given by the same parameterization in a common for a chosen na–map coordinate system on U and $\underline{U}$. This condition holds for vectors $\underline{v}^\alpha_{(1)} = v\underline{D}v^\alpha$ and $v^\alpha_{(2)} = v\underline{D}v^\alpha_{(1)}$ satisfying equality

$$\underline{v}^\alpha_{(2)} = \underline{a}(\eta) v^\alpha + \underline{b}(\eta) \underline{v}^\alpha_{(1)} \tag{3.5}$$

for some scalar functions $\underline{a}(\eta)$ and $\underline{b}(\eta)$ (see Definitions 3.2 and 3.3). Putting splittings (3.2) and (3.3) into expressions for $\underline{v}^\alpha_{(1)}$ and $\underline{v}^\alpha_{(2)}$ in (3.5) we obtain:

$$v^\beta v^\gamma v^\delta (D_\beta P^\alpha_{.\gamma\delta} + P^\alpha_{.\beta\tau} P^\tau_{.\gamma\delta} + Q^\alpha_{.\beta\tau} P^\tau_{.\gamma\delta}) = b v^\gamma v^\delta P^\alpha_{.\gamma\delta} + a v^\alpha, \tag{3.6}$$

where

$$b(\eta, v) = \underline{b} - 3\rho, \qquad \text{and} \qquad a(\eta, v) = \underline{a} + \underline{b}\rho - v^b \partial_b \rho - \rho^2 \tag{3.7}$$

are called the deformation parameters of na–maps.

The algebraic equations for the deformation of torsion $Q^\alpha_{.\beta\tau}$ should be written as the compatibility conditions for a given nonmetricity tensor $\underline{K}_{\alpha\beta\gamma}$ on $\underline{\xi}$ (or as the metricity conditions if d–connection $\underline{D}_\alpha$ on $\underline{\xi}$ is required to be metric) :

$$D_\alpha G_{\beta\gamma} - P^\delta_{.\alpha(\beta} G_{\gamma)\delta} - \underline{K}_{\alpha\beta\gamma} = Q^\delta_{.\alpha(\beta} G_{\gamma)\delta}, \tag{3.8}$$

where ( ) denotes the symmetric alternation.

So, we have proved this

Theorem 3.1. *The na–maps from la–space ξ to la–space $\underline{\xi}$ with a fixed common nonlinear connection $N^a_j(u) = \underline{N}^a_j(u)$ and given d–connections, $\Gamma^{\alpha}_{.\beta\gamma}$ on ξ and $\underline{\Gamma}^{\alpha}_{.\beta\gamma}$ on $\underline{\xi}$ are locally parametrized by the solutions of equations (3.6) and (3.8) for every point u^{α} and direction v^{α} on $U\subset\xi$.*

We call (3.6) and (3.8) the basic equations for na–maps of la–spaces. They generalize the corresponding equations [Sinyukov 1979] for isotropic spaces provided with symmetric affine connection structure.

III.2 Classification of Nearly Autoparallel Maps of LA–Spaces

Na–maps are classed on possible polynomial parametrizations on variables v^{α} of deformations parameters a and b (see (3.6) and (3.7)).

Theorem 3.2. *There are four classes of na–maps characterized by corresponding deformation parameters and tensors and basic equations:*
0 . for $na_{(0)}$–maps, $\pi_{(0)}$–maps,

$$P^{\alpha}_{\beta\gamma}(u) = \psi_{(\beta}\delta^{\alpha}_{\gamma)}$$

(δ^{α}_{β} is Kronecker symbol and $\psi_{\beta} = \psi_{\beta}(u)$ is a covariant vector d–field) ;

1 . for $na_{(1)}$–maps

$$a(u,v) = a_{\alpha\beta}(u)v^{\alpha}v^{\beta}, \quad b(u,v) = b_{\alpha}(u)v^{\alpha}$$

and $P^{\alpha}_{.\beta\gamma}(u)$ is the solution of equations

$$D_{(\alpha}P^{\delta}_{.\beta\gamma)} + P^{\tau}_{(\alpha\beta}P^{\delta}_{.\gamma)\tau} - P^{\tau}_{(\alpha\beta}Q^{\delta}_{.\gamma)\tau} = b_{(\alpha}P^{\delta}_{.\beta\gamma)} + a_{(\alpha\beta}\delta^{\delta}_{\gamma)}; \tag{3.9}$$

2 . for $na_{(2)}$–maps

$$a(u,v) = a_{\beta}(u)v^{\beta}, \quad b(u,v) = \frac{b_{\alpha\beta}v^{\alpha}v^{\beta}}{\sigma_{\alpha}(u)v^{\alpha}}, \quad \sigma_{\alpha}v^{\alpha}\neq 0,$$

$$P^{\tau}_{.\alpha\beta}(u) = \psi_{(\alpha}\delta^{\tau}_{\beta)} + \sigma_{(\alpha}F^{\tau}_{\beta)}$$

and $F^{\alpha}_{\beta}(u)$ is the solution of equations

$$D_{(\gamma}F^{\alpha}_{\beta)} + F^{\alpha}_{\delta}F^{\delta}_{(\gamma}\sigma_{\beta)} - Q^{\alpha}_{.\tau(\beta}F^{\tau}_{\gamma)} = \mu_{(\beta}F^{\alpha}_{\gamma)} + \nu_{(\beta}\delta^{\alpha}_{\gamma)} \tag{3.10}$$

($\mu_{\beta}(u), \nu_{\beta}(u), \psi_{\alpha}(u), \sigma_{\alpha}(u)$ are covariant d–vectors) ;
3 . for $na_{(3)}$–maps

$$b(u,v) = \frac{\alpha_{\beta\gamma\delta}v^{\beta}v^{\gamma}v^{\delta}}{\sigma_{\alpha\beta}v^{\alpha}v^{\gamma}},$$

$$P^{\alpha}_{.\beta\gamma}(u) = \psi_{(\beta}\delta^{\alpha}_{\gamma)} + \sigma_{\beta\gamma}\varphi^{\alpha},$$

where φ^α *is the solution of equations*

$$D_\beta \varphi^\alpha = \nu \delta^\alpha_\beta + \mu_\beta \varphi^\alpha + \varphi^\gamma Q^\alpha_{.\gamma\delta}, \tag{3.11}$$

$\alpha_{\beta\gamma\delta}(u), \sigma_{\alpha\beta}(u), \psi_\beta(u), \nu(u)$ *and* $\mu_\beta(u)$ *are d–tensors.*

Proof. We sketch the proof respectively for every point in the theorem:

0 . It is easy to verify that a–parallel equations (3.4) on ξ transform into similar ones on $\underline{\xi}$ if and only if deformations (3.2) with deformation d–tensors of type $P^\alpha{}_{\beta\gamma}(u) = \psi_{(\beta}\delta^\alpha_{\gamma)}$ are considered.

1 . Using corresponding to $na_{(1)}$–maps parametrizations of $a(u,v)$ and $b(u,v)$ (see conditions of the theorem) for arbitrary $v^\alpha \neq 0$ on $U \in \xi$ and after a redefinition of deformation parameters we obtain that equations (3.6) hold if and only if $P^\alpha{}_{\beta\gamma}$ satisfies (3.3).

2 . In a similar manner we obtain basic $na_{(2)}$–map equations (3.10) from (3.6) by considering $na_{(2)}$–parametrizations of deformation parameters and d–tensor.

3 . For $na_{(3)}$–maps we mast take into consideration deformations of torsion (3.3) and introduce $na_{(3)}$–parametrizations for $b(u,v)$ and $P^\alpha{}_{\beta\gamma}$ into the basic na–equations (3.6). The last ones for $na_{(3)}$–maps are equivalent to equations (3.11) (with a corresponding redefinition of deformation parameters). ■

We point out that for $\pi_{(0)}$-maps we do not have differential equations on $P^\alpha_{.\beta\gamma}$ (in the isotropic case one considers a first order system of differential equations on metric [Sinyukov 1979]; we omit constructions with deformation of metric in this work).

To formulate invariant conditions for reciprocal na–maps (when every a-parallel on $\underline{\xi}$ is also transformed into na–parallel on ξ) it is convenient to introduce into consideration the curvature and Ricci tensors defined for auxiliary connection $\gamma^\alpha_{.\beta\gamma}$:

$$r^{.\delta}_{\alpha.\beta\tau} = \partial_{[\beta}\gamma^\delta_{.\tau]\alpha} + \gamma^\delta_{.\rho[\beta}\gamma^\rho_{.\tau]\alpha} + \gamma^\delta{}_{\alpha\phi} w^\phi{}_{\beta\tau}$$

and, respectively, $r_{\alpha\tau} = r^{.\gamma}_{\alpha.\gamma\tau}$, where [ ] denotes antisymmetric alternation of indices, and to define values:

$$^{(0)}T^\mu_{.\alpha\beta} = \Gamma^\mu_{.\alpha\beta} - T^\mu_{.\alpha\beta} - \frac{1}{(n+m+1)}(\delta^\mu_{(\alpha}\Gamma^\delta_{.\beta)\delta} - \delta^\mu_{(\alpha}T^\delta_{.\beta)\gamma}),$$

$$^{(0)}W^{\cdot\tau}_{\alpha\cdot\beta\gamma} = r^{\cdot\tau}_{\alpha\cdot\beta\gamma} + \frac{1}{n+m+1}[\gamma^\tau_{\cdot\varphi\tau}\delta^\tau_{(\alpha}w^\varphi{}_{\beta)\gamma} - (\delta^\tau_\alpha r_{[\gamma\beta]} + \delta^\tau_\gamma r_{[\alpha\beta]} - \delta^\tau_\beta r_{[\alpha\gamma]})] -$$

$$\frac{1}{(n+m+1)^2}[\delta^\tau_\alpha(2\gamma^\tau_{\cdot\varphi\tau}w^\varphi{}_{[\gamma\beta]} - \gamma^\tau_{\cdot\tau[\gamma}w^\varphi{}_{\beta]\varphi}) + \delta^\tau_\gamma(2\gamma^\tau_{\cdot\varphi\tau}w^\varphi{}_{\alpha\beta} - \gamma^\tau_{\cdot\alpha\tau}w^\varphi{}_{\beta\varphi}) -$$

$$\delta^\tau_\beta(2\gamma^\tau_{\cdot\varphi\tau}w^\varphi{}_{\alpha\gamma} - \gamma^\tau_{\cdot\alpha\tau}w^\varphi{}_{\gamma\varphi})],$$

$$
{}^{(3)}T^{\delta}_{.\alpha\beta} = \gamma^{\delta}_{.\alpha\beta} + \epsilon\varphi^{\tau(\gamma)} D_\beta q_\tau + \frac{1}{n+m}(\delta^{\gamma}_{\alpha} - \epsilon\varphi^{\delta} q_\alpha)[\gamma^{\tau}_{.\beta\tau} + \epsilon\varphi^{\tau(\gamma)} D_\beta q_\tau +
$$

$$
\frac{1}{n+m-1} q_\beta (\epsilon\varphi^\tau \gamma^{\lambda}_{.\tau\lambda} + \varphi^\lambda \varphi^{\tau(\gamma)} D_\tau q_\lambda)] - \frac{1}{n+m}(\delta^{\delta}_{\beta} - \epsilon\varphi^{\delta} q_\beta)[\gamma^{\tau}_{.\alpha\tau} + \epsilon\varphi^{\tau(\gamma)} D_\alpha q_\tau +
$$

$$
\frac{1}{n+m-1} q_\alpha (\epsilon\varphi^\tau \gamma^{\lambda}_{.\tau\lambda} + \varphi^\lambda \varphi^{\tau(\gamma)} D_\tau q_\lambda)],
$$

$$
{}^{(3)}W^{\alpha}.\beta\gamma\delta = \rho^{\cdot\alpha}_{\beta.\gamma\delta} + \epsilon\varphi^\alpha q_\tau \rho^{\cdot\tau}_{\beta.\gamma\delta} + (\delta^{\alpha}_{\delta} -
$$

$$
\epsilon\varphi^\alpha q_\delta) p_{\beta\gamma} - (\delta^{\alpha}_{\gamma} - \epsilon\varphi^\alpha q_\gamma) p_{\beta\delta} - (\delta^{\alpha}_{\beta} - \epsilon\varphi^\alpha q_\beta) p_{[\gamma\delta]},
$$

$$
(n+m-2) p_{\alpha\beta} = -\rho_{\alpha\beta} - \epsilon q_\tau \varphi^\gamma \rho^{\cdot\tau}_{\alpha.\beta\gamma} + \frac{1}{n+m}[\rho^{\cdot\tau}_{\tau.\beta\alpha} - \epsilon q_\tau \varphi^\gamma \rho^{\cdot\tau}_{\gamma.\beta\alpha} + \epsilon q_\beta \varphi^\tau \rho_{\alpha\tau} +
$$

$$
\epsilon q_\alpha(-\varphi^\gamma \rho^{\cdot\tau}_{\tau.\beta\gamma} + \epsilon q_\tau \varphi^\gamma \varphi^\delta \rho^{\cdot\tau}_{\gamma.\beta\delta}]),
$$

where $q_\alpha \varphi^\alpha = \epsilon = \pm 1$,

$$
\rho^{\alpha}{}_{\beta\gamma\delta} = r^{\cdot\alpha}_{\beta\cdot\gamma\delta} + \frac{1}{2}(\psi_{(\beta}\delta^{\alpha}_{\varphi)} + \sigma_{\beta\varphi}\varphi^\tau) w^{\varphi}{}_{\gamma\delta}
$$

(for a similar value on $\underline{\xi}$ we write $\underline{\rho}^{\alpha}_{\cdot\beta\gamma\delta} = \underline{r}^{\cdot\alpha}_{\beta\cdot\gamma\delta} - \frac{1}{2}(\psi_{(\beta}\delta^{\alpha}_{\varphi)} - \sigma_{\beta\varphi}\varphi^\tau) w^{\varphi}{}_{\gamma\delta}$) and $\rho_{\alpha\beta} = \rho^{\tau}_{\cdot\alpha\beta\tau}$.

Similar values, ${}^{(0)}\underline{T}^{\alpha}_{.\beta\gamma}, {}^{(0)}\underline{W}^{\nu}_{.\alpha\beta\gamma}, \hat{T}^{\alpha}_{.\beta\gamma}, \check{T}^{\alpha}_{.\beta\tau}, \hat{W}^{\delta}_{.\alpha\beta\gamma}, \check{W}^{\delta}_{.\alpha\beta\gamma}, {}^{(3)}\underline{T}^{\delta}_{.\alpha\beta}$, and ${}^{(3)}\underline{W}^{\alpha}_{.\beta\gamma\delta}$ are given, correspondingly, by auxiliary connections $\underline{\Gamma}^{\mu}_{.\alpha\beta}$,

$$
\star\gamma^{\alpha}_{.\beta\lambda} = \gamma^{\alpha}_{.\beta\lambda} + \epsilon F^{\alpha(\gamma)}_{\tau} D_{(\beta} F^{\tau}_{\lambda)}, \quad \check{\gamma}^{\alpha}_{.\beta\lambda} = \widetilde{\gamma}^{\alpha}_{.\beta\lambda} + \epsilon F^{\lambda}_{\tau} \widetilde{D}_{(\beta} F^{\tau}_{\lambda)},
$$

$$
\widetilde{\gamma}^{\alpha}_{.\beta\tau} = \gamma^{\alpha}_{.\beta\tau} + \sigma_{(\beta} F^{\alpha}_{\tau)}, \quad \hat{\gamma}^{\alpha}_{.\beta\lambda} = \star\gamma^{\alpha}_{.\beta\lambda} + \widetilde{\sigma}_{(\beta}\delta^{\alpha}_{\lambda)},
$$

where $\widetilde{\sigma}_\beta = \sigma_\alpha F^{\alpha}_{\beta}$.

Theorem 3.3. *Four classes of reciprocal na–maps of la–spaces are characterized by corresponding invariant criterions:*

0 . for a–maps ${}^{(0)}T^{\mu}_{.\alpha\beta} = {}^{(0)}\underline{T}^{\mu}_{.\alpha\beta}$,

$$
{}^{(0)}W^{\delta}_{.\alpha\beta\gamma} = {}^{(0)}\underline{W}^{\delta}_{.\alpha\beta\gamma}; \tag{3.12}
$$

1 . for $na_{(1)}$–maps

$$
3({}^{(\gamma)}D_\lambda P^{\delta}_{.\alpha\beta} + P^{\delta}_{.\tau\lambda} P^{\tau}_{.\alpha\beta}) = r^{\cdot\delta}_{(\alpha.\beta)\lambda} - \underline{r}^{\cdot\delta}_{(\alpha.\beta)\lambda} + \tag{3.13}
$$

$$
[T^{\delta}_{.\tau(\alpha} P^{\tau}_{.\beta\lambda)} + Q^{\delta}_{.\tau(\alpha} P^{\tau}_{.\beta\lambda)} + b_{(\alpha} P^{\delta}_{.\beta\lambda)} + \delta^{\delta}_{(\alpha} a_{\beta\lambda)}];
$$

2 . for $na_{(2)}$*–maps* $\hat{T}^{\alpha}_{.\beta\tau} = \star T^{\alpha}_{.\beta\tau}$,

$$\hat{W}^{\delta}_{.\alpha\beta\gamma} = \star W^{\delta}_{.\alpha\beta\gamma}; \tag{3.14}$$

3 . for $na_{(3)}$*–maps* ${}^{(3)}T^{\alpha}_{.\beta\gamma} = {}^{(3)}\underline{T}^{\alpha}_{.\beta\gamma}$,

$${}^{(3)}W^{\alpha}_{.\beta\gamma\delta} = {}^{(3)}\underline{W}^{\alpha}_{.\beta\gamma\delta}. \tag{3.15}$$

Proof.

0 . Let us prove that a–invariant conditions (3.12) hold. Deformations of d–connections of type

$${}^{(0)}\underline{\gamma}^{\mu}_{\cdot\alpha\beta} = \gamma^{\mu}{}_{\alpha\beta} + \psi_{(\alpha}\delta^{\mu}_{\beta)} \tag{3.16}$$

define a–applications. Contracting indices μ and β we can write

$$\psi_{\alpha} = \frac{1}{m+n+1}(\underline{\gamma}^{\beta}{}_{\alpha\beta} - \gamma^{\beta}{}_{\alpha\beta}). \tag{3.17}$$

Introducing d–vector ψ_{α} into previous relation and expressing

$$\gamma^{\alpha}{}_{\beta\tau} = -T^{\alpha}{}_{\beta\tau} + \Gamma^{\alpha}{}_{\beta\tau}$$

and similarly for underlined values we obtain the first invariant conditions from (3.12).

Putting deformation (3.16) into the formula for

$$\underline{r}^{\cdot\tau}_{\alpha\cdot\beta\gamma} \quad \text{and} \quad \underline{r}_{\alpha\beta} = \underline{r}^{\cdot\tau}_{\alpha\tau\beta\tau}$$

we obtain respectively relations

$$\underline{r}^{\cdot\tau}_{\alpha\cdot\beta\gamma} - r^{\cdot\tau}_{\alpha\cdot\beta\gamma} = \delta^{\tau}_{\alpha}\psi_{[\gamma\beta]} + \psi_{\alpha[\beta}\delta^{\tau}_{\gamma]} + \delta^{\tau}_{(\alpha}\psi_{\varphi)}w^{\varphi}{}_{\beta\gamma} \tag{3.18}$$

and

$$\underline{r}_{\alpha\beta} - r_{\alpha\beta} = \psi_{[\alpha\beta]} + (n+m-1)\psi_{\alpha\beta} + \psi_{\varphi}w^{\varphi}{}_{\beta\alpha} + \psi_{\alpha}w^{\varphi}{}_{\beta\varphi}, \tag{3.19}$$

where

$$\psi_{\alpha\beta} = {}^{(\gamma)}D_{\beta}\psi_{\alpha} - \psi_{\alpha}\psi_{\beta}.$$

Putting (3.16) into (3.19) we can express $\psi_{[\alpha\beta]}$ as

$$\begin{aligned} \psi_{[\alpha\beta]} = {} & \frac{1}{n+m+1}[\underline{r}_{[\alpha\beta]} + \frac{2}{n+m+1}\underline{\gamma}^{\tau}_{\cdot\varphi\tau}w^{\varphi}{}_{[\alpha\beta]} - \frac{1}{n+m+1}\underline{\gamma}^{\tau}_{\cdot\tau[\alpha}w^{\varphi}{}_{\beta]\varphi}] - \\ & \frac{1}{n+m+1}[r_{[\alpha\beta]} + \frac{2}{n+m+1}\gamma^{\tau}{}_{\varphi\tau}w^{\varphi}{}_{[\alpha\beta]} - \frac{1}{n+m+1}\gamma^{\tau}{}_{\tau[\alpha}w^{\varphi}{}_{\beta]\varphi}]. \end{aligned} \tag{3.20}$$

To simplify our consideration we can choose an a–transform, parametrized by corresponding ψ–vector from (3.16), (or fix a local coordinate cart) the antisymmetrized relations (3.20) to be satisfied by d–tensor

(3.21)

$$\psi_{\alpha\beta} = \frac{1}{n+m+1}[\underline{r}_{\alpha\beta} + \frac{2}{n+m+1}\underline{\gamma}^{\tau}_{\cdot\varphi\tau}w^{\varphi}{}_{\alpha\beta} - \frac{1}{n+m+1}\underline{\gamma}^{\tau}_{\cdot\alpha\tau}w^{\varphi}{}_{\beta\varphi} - r_{\alpha\beta} - \frac{2}{n+m+1}\gamma^{\tau}{}_{\varphi\tau}w^{\varphi}{}_{\alpha\beta} + \frac{1}{n+m+1}\gamma^{\tau}{}_{\alpha\tau}w^{\varphi}{}_{\beta\varphi}]$$

Introducing expressions (3.16),(3.20) and (3.21) into deformation of curvature (3.17) we obtain the second conditions (3.12) of a–map invariance:

$$^{(0)}W^{\cdot\delta}_{\alpha\cdot\beta\gamma} = {}^{(0)}\underline{W}^{\cdot\delta}_{\alpha\cdot\beta\gamma},$$

where the Weyl d–tensor on $\underline{\xi}$ (the extension of the usual one for geodesic maps on (pseudo)–Riemannian spaces to the case of v–bundles provided with N–connection structure) is defined as

$$^{(0)}\underline{W}^{\cdot\tau}_{\alpha\cdot\beta\gamma} = \underline{r}^{\cdot\tau}_{\alpha\cdot\beta\gamma} + \frac{1}{n+m+1}[\underline{\gamma}^{\tau}_{\cdot\varphi\tau}\delta^{\tau}_{(\alpha}w^{\varphi}{}_{\beta)\gamma} - (\delta^{\tau}_{\alpha}\underline{r}_{[\gamma\beta]} + \delta^{\tau}_{\gamma}\underline{r}_{[\alpha\beta]} - \delta^{\tau}_{\beta}\underline{r}_{[\alpha\gamma]})] -$$

$$\frac{1}{(n+m+1)^2}[\delta^{\tau}_{\alpha}(2\underline{\gamma}^{\tau}_{\cdot\varphi\tau}w^{\varphi}{}_{[\gamma\beta]} - \underline{\gamma}^{\tau}_{\cdot\tau[\gamma}w^{\varphi}{}_{\beta]\varphi}) + \delta^{\tau}_{\gamma}(2\underline{\gamma}^{\tau}_{\cdot\varphi\tau}w^{\varphi}{}_{\alpha\beta} - \underline{\gamma}^{\tau}_{\cdot\alpha\tau}w^{\varphi}{}_{\beta\varphi}) -$$

$$\delta^{\tau}_{\beta}(2\underline{\gamma}^{\tau}_{\cdot\varphi\tau}w^{\varphi}{}_{\alpha\gamma} - \underline{\gamma}^{\tau}_{\cdot\alpha\tau}w^{\varphi}{}_{\gamma\varphi})]$$

The formula for $^{(0)}W^{\cdot\tau}_{\alpha\cdot\beta\gamma}$ written similarly with respect to non–underlined values is presented in subsection 1.1.2 .

1 . To obtain $na_{(1)}$–invariant conditions we rewrite $na_{(1)}$–equations (3.9) as to consider in explicit form covariant derivation $^{(\gamma)}D$ and deformations (3.2) and (3.3):

(3.22)

$$2({}^{(\gamma)}D_{\alpha}P^{\delta}{}_{\beta\gamma} + {}^{(\gamma)}D_{\beta}P^{\delta}{}_{\alpha\gamma} + {}^{(\gamma)}D_{\gamma}P^{\delta}{}_{\alpha\beta} + P^{\delta}{}_{\tau\alpha}P^{\tau}{}_{\beta\gamma} + P^{\delta}{}_{\tau\beta}P^{\tau}{}_{\alpha\gamma} + P^{\delta}{}_{\tau\gamma}P^{\tau}{}_{\alpha\beta}) = T^{\delta}{}_{\tau(\alpha}P^{\tau}{}_{\beta\gamma)} + H^{\delta}{}_{\tau(\alpha}P^{\tau}{}_{\beta\gamma)} + b_{(\alpha}P^{\delta}{}_{\beta\gamma)} + a_{(\alpha\beta}\delta^{\delta}_{\gamma)}.$$

Alternating the first two indices in (3.22) we have

$$2(\underline{r}^{\cdot\delta}_{(\alpha\cdot\beta)\gamma} - r^{\cdot\delta}_{(\alpha\cdot\beta)\gamma}) = 2({}^{(\gamma)}D_{\alpha}P^{\delta}{}_{\beta\gamma} +$$

$$^{(\gamma)}D_{\beta}P^{\delta}{}_{\alpha\gamma} - 2{}^{(\gamma)}D_{\gamma}P^{\delta}{}_{\alpha\beta} + P^{\delta}{}_{\tau\alpha}P^{\tau}{}_{\beta\gamma} + P^{\delta}{}_{\tau\beta}P^{\tau}{}_{\alpha\gamma} - 2P^{\delta}{}_{\tau\gamma}P^{\tau}{}_{\alpha\beta}).$$

Substituting the last expression from (3.22) and rescalling the deformation parameters and d–tensors we obtain the conditions (3.9).

2 . Now we prove the invariant conditions for $na_{(2)}$–maps satisfying conditions

$$\epsilon \neq 0 \quad \text{and} \quad \epsilon - F^{\alpha}_{\beta}F^{\beta}_{\alpha} \neq 0$$

Let define the auxiliary d–connection

$$\tilde{\gamma}^{\alpha}_{\cdot\beta\tau} = \underline{\gamma}^{\alpha}_{\cdot\beta\tau} - \psi_{(\beta}\delta^{\alpha}_{\tau)} = \gamma^{\alpha}{}_{\beta\tau} + \sigma_{(\beta}F^{\alpha}_{\tau)} \tag{3.23}$$

and write

$$\tilde{D}_{\gamma} = {}^{(\gamma)}D_{\gamma}F^{\alpha}_{\beta} + \tilde{\sigma}_{\gamma}F^{\alpha}_{\beta} - \epsilon\sigma_{\beta}\delta^{\alpha}_{\gamma},$$

where $\tilde{\sigma}_{\beta} = \sigma_{\alpha}F^{\alpha}_{\beta}$, or, as a consequence from the last equality,

$$\sigma_{(\alpha}F^{\tau}_{\beta)} = \epsilon F^{\tau}_{\lambda}({}^{(\gamma)}D_{(\alpha}F^{\alpha}_{\beta)} - \tilde{D}_{(\alpha}F^{\lambda}_{\beta)}) + \tilde{\sigma}_{(}\alpha\delta^{\tau}_{\beta)}.$$

Introducing auxiliary connections

$$\star\gamma^{\alpha}_{\cdot\beta\lambda} = \gamma^{\alpha}_{\cdot\beta\lambda} + \epsilon F^{\alpha}_{\tau}{}^{(\gamma)}D_{(\beta}F^{\tau}_{\lambda)}$$

and

$$\check{\gamma}^{\alpha}_{\cdot\beta\lambda} = \tilde{\gamma}^{\alpha}_{\cdot\beta\lambda} + \epsilon F^{\alpha}_{\tau}\tilde{D}_{(\beta}F^{\tau}_{\lambda)}$$

we can express deformation (3.23) in a form characteristic for a–maps:

$$\hat{\gamma}^{\alpha}_{\cdot\beta\gamma} = \star\gamma^{\alpha}_{\cdot\beta\gamma} + \tilde{\sigma}_{(\beta}\delta^{\alpha}_{\lambda)}. \tag{3.24}$$

Now it's obvious that $na_{(2)}$–invariant conditions (3.24) are equivalent with a–invariant conditions (3.12) written for d–connection (3.24). As a matter of principle we can write formulas for such $na_{(2)}$–invariants in terms of "underlined" and "non–underlined" values by expressing consequently all used auxiliary connections as deformations of "prime" connections on ξ and "final" connections on $\underline{\xi}$. We omit such tedious calculations in this work.

3 . Finally, we prove the last statement, for $na_{(3)}$–maps, of the theorem. Let

$$q_{\alpha}\varphi^{\alpha} = e = \pm 1, \tag{3.25}$$

where φ^{α} is contained in

$$\underline{\gamma}^{\alpha}_{\cdot\beta\gamma} = \gamma^{\alpha}{}_{\beta\gamma} + \psi_{(\beta}\delta^{\alpha}_{\gamma)} + \sigma_{\beta\gamma}\varphi^{\alpha}. \tag{3.26}$$

Acting with operator ${}^{(\gamma)}\underline{D}_{\beta}$ on (3.25) we write

$${}^{(\gamma)}\underline{D}_{\beta}q_{\alpha} = {}^{(\gamma)}D_{\beta}q_{\alpha} - \psi_{(\alpha}q_{\beta)} - e\sigma_{\alpha\beta}. \tag{3.27}$$

Contracting (3.27) with φ^{α} we can express

$$e\varphi^{\alpha}\sigma_{\alpha\beta} = \varphi^{\alpha}({}^{(\gamma)}D_{\beta}q_{\alpha} - {}^{(\gamma)}\underline{D}_{\beta}q_{\alpha}) - \varphi_{\alpha}q^{\alpha}q_{\beta} - e\psi_{\beta}.$$

Putting the last formula in (3.26) contracted on indices α and γ we obtain

$$(n+m)\psi_{\beta} = \underline{\gamma}^{\alpha}_{\cdot\alpha\beta} - \gamma^{\alpha}{}_{\alpha\beta} + e\psi_{\alpha}\varphi^{\alpha}q_{\beta} + e\varphi^{\alpha}\varphi^{\beta}({}^{(\gamma)}\underline{D}_{\beta} - {}^{(\gamma)}D_{\beta}). \tag{3.28}$$

From these relations, taking into consideration (3.25), we have

$$(n+m-1)\psi_\alpha\varphi^\alpha = \varphi^\alpha(\underline{\gamma}^{\alpha}_{\cdot\alpha\beta} - \gamma^{\alpha}{}_{\alpha\beta}) + e\varphi^\alpha\varphi^\beta({}^{(\gamma)}\underline{D}_\beta q_\alpha - {}^{(\gamma)}D_\beta q_\alpha)$$

Using the equalities and identities (3.27) and (3.28) we can express deformations (3.26) as the first $na_{(3)}$–invariant conditions from (3.15).

To prove the second class of $na_{(3)}$–invariant conditions we introduce two additional d–tensors:

$$\rho^{\alpha}{}_{\beta\gamma\delta} = r^{\cdot\alpha}_{\beta\cdot\gamma\delta} + \frac{1}{2}(\psi_{(\beta}\delta^{\alpha}_{\varphi)} + \sigma_{\beta\varphi}\varphi^\tau)w^{\varphi}{}_{\gamma\delta}$$

and

$$\underline{\rho}^{\alpha}_{\cdot\beta\gamma\delta} = \underline{r}^{\cdot\alpha}_{\beta\cdot\gamma\delta} - \frac{1}{2}(\psi_{(\beta}\delta^{\alpha}_{\varphi)} - \sigma_{\beta\varphi}\varphi^\tau)w^{\varphi}{}_{\gamma\delta}. \tag{3.29}$$

Using deformation (3.26) and (3.29) we write relation

$$\tilde{\sigma}^{\alpha}_{\cdot\beta\gamma\delta} = \underline{\rho}^{\alpha}_{\cdot\beta\gamma\delta} - \rho^{\alpha}_{\cdot\beta\gamma\delta} = \psi_{\beta[\delta}\delta^{\alpha}_{\gamma]} - \psi_{[\gamma\delta]}\delta^{\alpha}_{\beta} - \sigma_{\beta\gamma\delta}\varphi^\alpha, \tag{3.30}$$

where

$$\psi_{\alpha\beta} = {}^{(\gamma)}D_\beta\psi_\alpha + \psi_\alpha\psi_\beta - (\nu + \varphi^\tau\psi_\tau)\sigma_{\alpha\beta},$$

and

$$\sigma_{\alpha\beta\gamma} = {}^{(\gamma)}D_{[\gamma}\sigma_{\beta]\alpha} + \mu_{[\gamma}\sigma_{\beta]\alpha} - \sigma_{\alpha[\gamma}\sigma_{\beta]\tau}\varphi^\tau.$$

Let multiply (3.30) on q_α and write (taking into account relations (3.25)) the relation

$$e\sigma_{\alpha\beta\gamma} = -q_\tau\tilde{\sigma}^{\tau}_{\cdot\alpha\beta\delta} + \psi_{\alpha[\beta}q_{\gamma]} - \psi_{[\beta\gamma]}q_\alpha. \tag{3.31}$$

The next step is to express $\psi_{\alpha\beta}$ trough d–objects on ξ. To do this we contract indices α and β in (3.30) and obtain

$$(n+m)\psi_{[\alpha\beta]} = -\sigma^{\tau}_{\cdot\tau\alpha\beta} + eq_\tau\varphi^\lambda\sigma^{\tau}_{\cdot\lambda\alpha\beta} - e\tilde{\psi}_{[\alpha}\tilde{\psi}_{\beta]}.$$

Then contracting indices α and δ in (3.30) and using (3.31) we write

$$(n+m-2)\psi_{\alpha\beta} = \tilde{\sigma}^{\tau}_{\cdot\alpha\beta\tau} - eq_\tau\varphi^\lambda\tilde{\sigma}^{\tau}_{\cdot\alpha\beta\lambda} + \psi_{[\beta\alpha]} + e(\tilde{\psi}_\beta q_\alpha - \hat{\psi}_{(\alpha}q_{\beta)}), \tag{3.32}$$

where $\hat{\psi}_\alpha = \varphi^\tau\psi_{\alpha\tau}$. If the both parts of (3.32) are contracted with φ^α, it results that

$$(n+m-2)\tilde{\psi}_\alpha = \varphi^\tau\sigma^{\lambda}_{\cdot\tau\alpha\lambda} - eq_\tau\varphi^\lambda\varphi^\delta\sigma^{\tau}_{\lambda\alpha\delta} - eq_\alpha,$$

and, in consequence of $\sigma^{\alpha}_{\beta(\gamma\delta)} = 0$, we have

$$(n+m-1)\varphi = \varphi^\beta\varphi^\gamma\sigma^{\alpha}_{\cdot\beta\gamma\alpha}.$$

By using the last expressions we can write

$$(3.33)\quad (n+m-2)\underline{\psi}_{\alpha} = \varphi^{\tau}\sigma^{\lambda}_{\cdot\tau\alpha\lambda} - eq_{\tau}\varphi^{\lambda}\varphi^{\delta}\sigma^{\tau}_{\cdot\lambda\alpha\delta} - e(n+m-1)^{-1}q_{\alpha}\varphi^{\tau}\varphi^{\lambda}\sigma^{\delta}_{\cdot\tau\lambda\delta}.$$

Contracting (3.32) with φ^{β} we have

$$(n+m)\hat{\psi}_{\alpha} = \varphi^{\tau}\sigma^{\lambda}_{\cdot\alpha\tau\lambda} + \tilde{\psi}_{\alpha}$$

and taking into consideration (3.33) we can express $\hat{\psi}_{\alpha}$ through $\sigma^{\alpha}_{\cdot\beta\gamma\delta}$.

As a consequence of (3.31)–(3.33) we obtain this formulas for d–tensor $\psi_{\alpha\beta}$:

$$(n+m-2)\psi_{\alpha\beta} = \sigma^{\tau}_{\cdot\alpha\beta\tau} - eq_{\tau}\varphi^{\lambda}\sigma^{\tau}_{\cdot\alpha\beta\lambda} +$$

$$\frac{1}{n+m}\{-\sigma^{\tau}_{\cdot\tau\beta\alpha} + eq_{\tau}\varphi^{\lambda}\sigma^{\tau}_{\cdot\lambda\beta\alpha} - q_{\beta}(e\varphi^{\tau}\sigma^{\lambda}_{\cdot\alpha\tau\lambda} - q_{\tau}\varphi^{\lambda}\varphi^{\delta}\sigma^{\tau}_{\cdot\alpha\lambda\delta}) +$$

$$eq_{\alpha}[\varphi^{\lambda}\sigma^{\tau}_{\cdot\tau\beta\lambda} - eq_{\tau}\varphi^{\lambda}\varphi^{\delta}\sigma^{\tau}_{\cdot\lambda\beta\delta} - \frac{e}{n+m-1}q_{\beta}(\varphi^{\tau}\varphi^{\lambda}\sigma^{\delta}_{\cdot\tau\gamma\delta} - eq_{\tau}\varphi^{\lambda}\varphi^{\delta}\varphi^{\varepsilon}\sigma^{\tau}_{\cdot\lambda\delta\varepsilon})]\}.$$

Finally, putting the last formula and (3.31) into (3.30) and after a rearrangement of terms we obtain the second group of $na_{(3)}$-invariant conditions (3.15). If necessary we can rewrite these conditions in terms of geometrical objects on ξ and $\underline{\xi}$. To do this we mast introduce splittings (3.29) into (3.15). ■

For the particular case of $na_{(3)}$–maps when

$$\psi_{\alpha} = 0, \varphi_{\alpha} = g_{\alpha\beta}\varphi^{\beta} = \frac{\delta}{\delta u^{\alpha}}(\ln\Omega), \Omega(u) > 0$$

and

$$\sigma_{\alpha\beta} = g_{\alpha\beta}$$

we define a subclass of conformal transforms $\underline{g}_{\alpha\beta}(u) = \Omega^{2}(u)g_{\alpha\beta}$ which, in consequence of the fact that d–vector φ_{α} must satisfy equations (3.11), generalizes the class of concircular transforms (see [Sinyukov 1979] for references and details on concircular mappings of Riemannaian spaces) .

We emphasize that basic na–equations (3.9)–(3.11) are systems of first order partial differential equations. The study of their geometrical properties and definition of integral varieties, general and particular solutions are possible by using the formalism of Pffaf systems [Cartan 1945]. Here we point out that by using algebraic methods we can always verify if systems of na–equations of type (3.9)–(3.11) are, or not, involute, even to find their explicit solutions it is a difficult task (see more detailed considerations for isotropic ng–maps in [Sinyukov 1979] and, on language of Pffaf systems for na–maps, in [Vacaru 1994]). We can also formulate the Cauchy problem for na–equations on ξ and choose deformation parameters (3.7) as to make involute mentioned equations for the case of maps to a given background space $\underline{\xi}$. If a solution, for example, of $na_{(1)}$–map equations exists, we say that space ξ is $na_{(1)}$–projective to space $\underline{\xi}$. In general, we have to introduce chains of na–maps in

order to obtain involute systems of equations for maps (superpositions of na-maps) from ξ to $\underline{\xi}$:

$$U \overset{ng<i_1>}{\longrightarrow} U_{\underline{1}} \overset{ng<i_2>}{\longrightarrow} \dots \overset{ng<i_{k-1}>}{\longrightarrow} U_{\underline{k-1}} \overset{ng<i_k>}{\longrightarrow} \underline{U}$$

where $U \subset \xi, U_{\underline{1}} \subset \xi_{\underline{1}}, \dots, U_{k-1} \subset \xi_{k-1}, \underline{U} \subset \xi_k$ with corresponding splittings of auxiliary symmetric connections

$$\underline{\gamma}^{\alpha}_{.\beta\gamma} =_{<i_1>} P^{\alpha}_{.\beta\gamma} +_{<i_2>} P^{\alpha}_{.\beta\gamma} + \dots +_{<i_k>} P^{\alpha}_{.\beta\gamma}$$

and torsion

$$\underline{T}^{\alpha}_{.\beta\gamma} = T^{\alpha}_{.\beta\gamma} +_{<i_1>} Q^{\alpha}_{.\beta\gamma} +_{<i_2>} Q^{\alpha}_{.\beta\gamma} + \dots +_{<i_k>} Q^{\alpha}_{.\beta\gamma}$$

where cumulative indices $< i_1 >= 0, 1, 2, 3,$ denote possible types of na–maps.

Definition 3.4. *Space ξ is nearly conformally projective to space $\underline{\xi}$, nc : $\xi \to \underline{\xi}$, if there is a finite chain of na–maps from ξ to $\underline{\xi}$.*

For nearly conformal maps we formulate :

Theorem 3.4. *For every fixed triples $(N^a_j, \Gamma^{\alpha}_{.\beta\gamma}, U \subset \xi$ and $(N^a_j, \underline{\Gamma}^{\alpha}_{.\beta\gamma}, \underline{U} \subset \underline{\xi})$, components of nonlinear connection, d–connection and d–metric being of class $C^r(U), C^r(\underline{U})$, $r > 3$, there is a finite chain of na–maps $nc : U \to \underline{U}$.*

Proof is similar to that for isotropic maps [Vacaru 1994] (we have to introduce a finite number of na-maps with corresponding components of deformation parameters and deformation tensors in order to transform step by step coefficients of d-connection $\Gamma^{\alpha}_{\gamma\delta}$ into $\underline{\Gamma}^{\alpha}_{\beta\gamma}$).

Now we introduce the concept of the Category of la–spaces, $\mathcal{C}(\xi)$. The elements of $\mathcal{C}(\xi)$ consist from $Ob\mathcal{C}(\xi) = \{\xi, \xi_{<i_1>}, \xi_{<i_2>}, \dots,\}$ being la–spaces, for simplicity in this work, having common N–connection structures, and $Mor\mathcal{C}(\xi) = \{nc(\xi_{<i_1>}, \xi_{<i_2>})\}$ being chains of na–maps interrelating la–spaces. We point out that we can consider equivalent models of physical theories on every object of $\mathcal{C}(\xi)$ (see details for isotropic gravitational models in [Petrov 1970] and [Vacaru 1994] and anisotropic gravity in [Vacau and Ostaf 1994, 1996a, 1996b]). One of the main purposes of this chapter is to develop a d–tensor and variational formalism on $\mathcal{C}(\xi)$, i.e. on la–multispaces, interrelated with nc–maps. Taking into account the distinguished character of geometrical objects on la–spaces we call tensors on $\mathcal{C}(\xi)$ as distinguished tensors on la–space Category, or dc–tensors.

Finally, we emphasize that presented in that section definitions and theorems can be generalized for v–bundles with arbitrary given structures of nonlinear connection, linear d–connection and metric structures. We omit such combersome calculations connected with deformation of all basic N–connection, d–connection and d–metric structures.

III.3 Nearly Autoparallel Tensor–Integral on LA–Spaces

The aim of this section is to define tensor integration not only for bitensors, objects defined on the same curved space, but for dc–tensors, defined on two spaces, ξ and $\underline{\xi}$, even it is necessary on la–multispaces.

Let $T_u\xi$ and $T_{\underline{u}}\underline{\xi}$ be tangent spaces in corresponding points $u\in U\subset\xi$ and $\underline{u}\in\underline{U}\subset\underline{\xi}$ and, respectively, $T_u^*\xi$ and $T_{\underline{u}}^*\underline{\xi}$ be their duals (in general, in this section we shall not consider that a common coordinatization is introduced for open regions U and $\underline{U}$). We call as the dc–tensors on the pair of spaces $(\xi,\underline{\xi}\ )$ the elements of distinguished tensor algebra

$$(\otimes_\alpha T_u\xi)\otimes(\otimes_\beta T_u^*\xi)\otimes(\otimes_\gamma T_{\underline{u}}\underline{\xi})\otimes(\otimes_\delta T_{\underline{u}}^*\underline{\xi})$$

defined over the space $\xi\otimes\underline{\xi}$, for a given $nc:\xi\to\underline{\xi}$.

We admit the convention that underlined and non–underlined indices refer, respectively, to the points $\underline{u}$ and u. Thus $Q^{\beta}_{.\underline{\alpha}}$, for instance, are the components of dc–tensor $Q\in T_u\xi\otimes T_{\underline{u}}\underline{\xi}$.

Now, we define the transport dc–tensors. Let open regions U and $\underline{U}$ be homeomorphic to sphere $\mathbb{R}^{2n}$ and introduce isomorphism $\mu_{u,\underline{u}}$ between $T_u\xi$ and $T_{\underline{u}}\underline{\xi}$ (given by map $nc:U\to\underline{U}$). We consider that for every d–vector $v^\alpha\in T_u\xi$ corresponds the vector $\mu_{u,\underline{u}}(v^\alpha)=v^{\underline{\alpha}}\in T_{\underline{u}}\underline{\xi}$, with components $v^{\underline{\alpha}}$ being linear functions of v^α:

$$v^{\underline{\alpha}}=h^{\underline{\alpha}}_{\alpha}(u,\underline{u})v^\alpha,\quad v_{\underline{\alpha}}=h^{\alpha}_{\underline{\alpha}}(\underline{u},u)v_\alpha,$$

where $h^{\alpha}_{\underline{\alpha}}(\underline{u},u)$ are the components of dc–tensor associated with $\mu^{-1}_{u,\underline{u}}$. In a similar manner we have

$$v^{\alpha}=h^{\alpha}_{\underline{\alpha}}(\underline{u},u)v^{\underline{\alpha}},\quad v_\alpha=h^{\underline{\alpha}}_{\alpha}(u,\underline{u})v_{\underline{\alpha}}.$$

In order to reconcile just presented definitions and to assure the identity for trivial maps $\xi\to\xi, u=\underline{u}$, the transport dc–tensors must satisfy conditions:

$$h^{\underline{\alpha}}_{\alpha}(u,\underline{u})h^{\beta}_{\underline{\alpha}}(\underline{u},u)=\delta^{\beta}_{\alpha}, h^{\underline{\alpha}}_{\alpha}(u,\underline{u})h^{\alpha}_{\underline{\beta}}(\underline{u},u)=\delta^{\underline{\alpha}}_{\underline{\beta}}$$

and $\lim_{(\underline{u}\to u)}h^{\underline{\alpha}}_{\alpha}(u,\underline{u})=\delta^{\underline{\alpha}}_{\alpha},\quad \lim_{(\underline{u}\to u)}h^{\alpha}_{\underline{\alpha}}(\underline{u},u)=\delta^{\alpha}_{\underline{\alpha}}.$

Let $\overline{S}_p\subset\overline{U}\subset\overline{\xi}$ is a homeomorphic to p-dimensional sphere and suggest that chains of na–maps are used to connect regions :

$$U\overset{nc_{(1)}}{\longrightarrow}\overline{S}_p\overset{nc_{(2)}}{\longrightarrow}\underline{U}.$$

Definition 3.5. *The tensor integral in $\overline{u}\in\overline{S}_p$ of a dc–tensor $N^{.\gamma.\underline{\kappa}}_{\varphi.\underline{\tau}.\overline{\alpha}_1\cdots\overline{\alpha}_p}\ (\overline{u},u)$, completely antisymmetric on the indices $\overline{\alpha}_1,\ldots,\overline{\alpha}_p$, over domain $\overline{S}_p$, is defined as*

$$N^{.\gamma.\underline{\kappa}}_{\varphi.\underline{\tau}}(\underline{u},u)=I^{\underline{U}}_{(\overline{S}_p)}N^{.\gamma.\overline{\kappa}}_{\varphi.\overline{\tau}.\overline{\alpha}_1\ldots\overline{\alpha}_p}(\overline{u},\underline{u})dS^{\overline{\alpha}_1\ldots\overline{\alpha}_p}= \tag{3.34}$$

$$\int_{(\overline{S}_p)}h^{\overline{\tau}}_{\underline{\tau}}(\underline{u},\overline{u})h^{\underline{\kappa}}_{\overline{\kappa}}(\overline{u},\underline{u})N^{.\gamma.\overline{\kappa}}_{\varphi.\overline{\tau}.\overline{\alpha}_1\cdots\overline{\alpha}_p}(\overline{u},u)d\overline{S}^{\overline{\alpha}_1\cdots\overline{\alpha}_p},$$

where $dS^{\overline{\alpha}_1\cdots\overline{\alpha}_p} = \delta u^{\overline{\alpha}_1}\wedge\cdots\wedge\delta u_p^{\overline{\alpha}}$.

Let suppose that transport dc–tensors $h^{\underline{\alpha}}_{\alpha}$ and $h^{\alpha}_{\underline{\alpha}}$ admit covariant derivations of order two and postulate existence of deformation dc–tensor $B^{\cdot\cdot\gamma}_{\alpha\beta}(u,\underline{u})$ satisfying relations

$$D_\alpha h^{\underline{\beta}}_{\beta}(u,\underline{u}) = B^{\cdot\cdot\gamma}_{\alpha\beta}(u,\underline{u})h^{\underline{\beta}}_{\gamma}(u,\underline{u}) \tag{3.35}$$

and, taking into account that $D_\alpha \delta^\beta_\gamma = 0$,

$$D_\alpha h^{\beta}_{\underline{\beta}}(\underline{u},u) = -B^{\cdot\cdot\beta}_{\alpha\gamma}(u,\underline{u})h^{\gamma}_{\underline{\beta}}(\underline{u},u).$$

By using formulas for torsion and, respectively, curvature of connection $\Gamma^\alpha_{\beta\gamma}$ we can calculate next commutators:

$$D_{[\alpha}D_{\beta]}h^{\underline{\gamma}}_{\gamma} = -(R^{\cdot\lambda}_{\gamma.\alpha\beta} + T^{\tau}_{.\alpha\beta}B^{\cdot\cdot\lambda}_{\tau\gamma})h^{\underline{\gamma}}_{\lambda}. \tag{3.36}$$

On the other hand from (3.35) one follows that

$$D_{[\alpha}D_{\beta]}h^{\underline{\gamma}}_{\gamma} = (D_{[\alpha}B^{\cdot\cdot\lambda}_{\beta]\gamma} + B^{\cdot\cdot\lambda}_{[\alpha|\tau|.}B^{\cdot\cdot\tau}_{\beta]\gamma.})h^{\underline{\gamma}}_{\lambda}, \tag{3.37}$$

where $|\tau|$ denotes that index τ is excluded from the action of antisymmetrization $[\quad]$. From (3.36) and (3.37) we obtain

$$D_{[\alpha}B^{\cdot\cdot\lambda}_{\beta]\gamma.} + B_{[\beta|\gamma|}B^{\cdot\cdot\lambda}_{\alpha]\tau} = (R^{\cdot\lambda}_{\gamma.\alpha\beta} + T^{\tau}_{.\alpha\beta}B^{\cdot\cdot\lambda}_{\tau\gamma}). \tag{3.38}$$

Let $\overline{S}_p$ be the boundary of $\overline{S}_{p-1}$. The Stoke's type formula for tensor–integral (3.34) is defined as

$$I_{\overline{S}_p} N^{\cdot\gamma.\overline{\kappa}}_{\varphi.\overline{\tau}.\overline{\alpha}_1\ldots\overline{\alpha}_p} dS^{\overline{\alpha}_1\ldots\overline{\alpha}_p} = I_{\overline{S}_{p+1}} {}^{\star(p)}\overline{D}_{[\overline{\gamma}|} N^{\cdot\gamma.\overline{\kappa}}_{\varphi.\overline{\tau}.|\overline{\alpha}_1\ldots\overline{\alpha}_p]} dS^{\overline{\gamma\alpha_1}\ldots\overline{\alpha}_p}, \tag{3.39}$$

where

$$\begin{aligned} {}^{\star(p)}D_{[\overline{\gamma}|} N^{\cdot\gamma.\overline{\kappa}}_{\varphi.\overline{\tau}.|\overline{\alpha}_1\ldots\overline{\alpha}_p]} &= D_{[\overline{\gamma}|} N^{\cdot\gamma.\overline{\kappa}}_{\varphi.\overline{\tau}.|\overline{\alpha}_1\ldots\overline{\alpha}_p]} + \\ & pT^{\overline{\epsilon}}_{.[\overline{\gamma\alpha_1}|} N^{\cdot\gamma.\overline{\kappa}}_{\varphi.\overline{\tau}.\overline{\epsilon}|\overline{\alpha}_2\ldots\overline{\alpha}_p]} - B^{\cdot\cdot\overline{\epsilon}}_{[\overline{\gamma}|\overline{\tau}} N^{\cdot\gamma.\overline{\kappa}}_{\varphi.\overline{\epsilon}.|\overline{\alpha}_1\ldots\overline{\alpha}_p]} + B^{\cdot\cdot\overline{\kappa}}_{[\overline{\gamma}|\overline{\epsilon}} N^{\cdot\gamma.\overline{\epsilon}}_{\varphi.\overline{\tau}.|\overline{\alpha}_1\ldots\overline{\alpha}_p]}. \end{aligned} \tag{3.40}$$

We define the dual element of the hypersurfaces element $dS^{j_1\ldots j_p}$ as

$$d\mathcal{S}_{\beta_1\ldots\beta_{q-p}} = \frac{1}{p!}\epsilon_{\beta_1\ldots\beta_{k-p}\alpha_1\ldots\alpha_p} dS^{\alpha_1\ldots\alpha_p}, \tag{3.41}$$

where $\epsilon_{\gamma_1\ldots\gamma_q}$ is completely antisymmetric on its indices and

$$\epsilon_{12\ldots(n+m)} = \sqrt{|G|}, G = det|G_{\alpha\beta}|,$$

$G_{\alpha\beta}$ is taken from (1.12). The dual of dc–tensor $N^{\cdot\gamma\overline{\kappa}}_{\varphi.\overline{\tau}.\overline{\alpha}_1\ldots\overline{\alpha}_p}$ is defined as the dc–tensor $\mathcal{N}^{\cdot\gamma.\overline{\kappa}\overline{\beta}_1\ldots\overline{\beta}_{n+m-p}}_{\varphi.\overline{\tau}}$ satisfying

$$N^{\cdot\gamma.\overline{\kappa}}_{\varphi.\overline{\tau}.\overline{\alpha}_1\ldots\overline{\alpha}_p} = \frac{1}{p!}\mathcal{N}^{\cdot\gamma.\overline{\kappa}\overline{\beta}_1\ldots\overline{\beta}_{n+m-p}}_{\varphi.\overline{\tau}}\epsilon_{\overline{\beta}_1\ldots\overline{\beta}_{n+m-p}\overline{\alpha}_1\ldots\overline{\alpha}_p}. \tag{3.42}$$

Using (3.16), (3.41) and (3.42) we can write

$$I_{\overline{S}_p} N^{\cdot\gamma.\overline{\kappa}}_{\varphi.\overline{\tau}.\overline{\alpha}_1\ldots\overline{\alpha}_p} dS^{\overline{\alpha}_1\ldots\overline{\alpha}_p} = \int_{\overline{S}_{p+1}} {}^{\overline{p}}D_{\overline{\gamma}}\mathcal{N}^{\cdot\gamma.\overline{\kappa}\overline{\beta}_1\ldots\overline{\beta}_{n+m-p-1}\overline{\gamma}}_{\varphi.\overline{\tau}} d\mathcal{S}_{\overline{\beta}_1\ldots\overline{\beta}_{n+m-p-1}}, \tag{3.43}$$

where

$${}^{\overline{p}}D_{\overline{\gamma}}\mathcal{N}^{\cdot\gamma.\overline{\kappa}\overline{\beta}_1\ldots\overline{\beta}_{n+m-p-1}\overline{\gamma}}_{\varphi.\overline{\tau}} =$$

$$\overline{D}_{\overline{\gamma}}\mathcal{N}^{\cdot\gamma.\overline{\kappa}\overline{\beta}_1\ldots\overline{\beta}_{n+m-p-1}\overline{\gamma}}_{\varphi.\overline{\tau}} + (-1)^{(n+m-p)}(n+m-p+1)T^{[\overline{\epsilon}}_{.\overline{\gamma}\overline{\epsilon}}\mathcal{N}^{\cdot|\gamma.\overline{\kappa}|\overline{\beta}_1\ldots\overline{\beta}_{n+m-p-1}]\overline{\gamma}}_{\varphi.\overline{\tau}} -$$

$$B^{..\overline{\epsilon}}_{\overline{\gamma}\overline{\tau}}\mathcal{N}^{\cdot\gamma.\overline{\kappa}\overline{\beta}_1\ldots\overline{\beta}_{n+m-p-1}\overline{\gamma}}_{\varphi.\overline{\epsilon}} + B^{..\overline{\kappa}}_{\overline{\gamma}\overline{\epsilon}}\mathcal{N}^{\cdot\gamma.\overline{\epsilon}\overline{\beta}_1\ldots\overline{\beta}_{n+m-p-1}\overline{\gamma}}_{\varphi.\overline{\tau}}.$$

To verify the equivalence of (3.42) and (3.43) we must take in consideration that

$$D_\gamma\epsilon_{\alpha_1\ldots\alpha_k} = 0 \text{ and } \epsilon_{\beta_1\ldots\beta_{n+m-p}\alpha_1\ldots\alpha_p}\epsilon^{\beta_1\ldots\beta_{n+m-p}\gamma_1\ldots\gamma_p} = p!(n+m-p)!\delta^{[\gamma_1}_{\alpha_1}\cdots\delta^{\gamma_p]}_{\alpha_p}.$$

The developed in this section tensor integration formalism will be used in the next section for definition of conservation laws on spaces with local anisotropy.

III.4 On Conservation Laws on La–Spaces

To define conservation laws on locally anisotropic spaces is a challenging task because of absence of global and local groups of automorphisms of such spaces. Our main idea is to use chains of na–maps from a given, called hereafter as the fundamental la–space to an auxiliary one with trivial curvatures and torsions admitting a global group of automorphisms. The aim of this section is to formulate conservation laws for la-gravitational fields by using dc–objects and tensor–integral values, na–maps and variational calculus on the Category of la–spaces.

III.4.1 Nonzero divergence of the energy–momentum d–tensor.

In work [Miron and Anastasiei 1994] it is pointed to this specific form of conservation laws of matter on la–spaces: They calculated the divergence of the energy–momentum d–tensor on la–space ξ,

$$D_\alpha E^\alpha_\beta = \frac{1}{\kappa_1}U_\alpha, \tag{3.44}$$

and concluded that d–vector

$$U_\alpha = \frac{1}{2}(G^{\beta\delta}R_\delta{}^\gamma{}_{\phi\beta}T^\phi_{\cdot\alpha\gamma} - G^{\beta\delta}R_\delta{}^\gamma{}_{\phi\alpha}T^\phi_{\cdot\beta\gamma} + R^\beta_\phi T^\phi_{\cdot\beta\alpha})$$

vanishes if and only if d–connection D is without torsion.

No wonder that conservation laws, in usual physical theories being a consequence of global (for usual gravity of local) automorphisms of the fundamental space–time, are more sophisticate on the spaces with local anisotropy. Here it is important to emphasize the multiconnection character of la–spaces. For example, for a d–metric (1.12) on ξ we can equivalently introduce another (see (1.23)) metric linear connection $\tilde{D}$. The Einstein equations

$$\tilde{R}_{\alpha\beta} - \frac{1}{2} G_{\alpha\beta} \tilde{R} = \kappa_1 \tilde{E}_{\alpha\beta} \tag{3.45}$$

constructed by using connection (3.23) have vanishing divergences

$$\tilde{D}^{\alpha}(\tilde{R}_{\alpha\beta} - \frac{1}{2} G_{\alpha\beta} \tilde{R}) = 0 \text{ and } \tilde{D}^{\alpha} \tilde{E}_{\alpha\beta} = 0,$$

similarly as those on (pseudo)Riemannian spaces. We conclude that by using the connection (1.23) we construct a model of la–gravity which looks like locally isotropic on the total space E. More general gravitational models with local anisotropy can be obtained by using deformations of connection $\tilde{\Gamma}^{\alpha}_{\cdot\beta\gamma}$,

$$\Gamma^{\alpha}{}_{\beta\gamma} = \tilde{\Gamma}^{\alpha}_{\cdot\beta\gamma} + P^{\alpha}{}_{\beta\gamma} + Q^{\alpha}{}_{\beta\gamma},$$

were, for simplicity, $\Gamma^{\alpha}{}_{\beta\gamma}$ is chosen to be also metric and satisfy Einstein equations (3.45). We can consider deformation d–tensors $P^{\alpha}{}_{\beta\gamma}$ generated (or not) by deformations of type (3.9)–(3.11) for na–maps. In this case d–vector U_{α} can be interpreted as a generic source of local anisotropy on ξ satisfying generalized conservation laws (3.44).

III.4.2 Deformation d–tensors and tensor–integral conservation laws.

From (3.34) we obtain a tensor integral on $\mathcal{C}(\xi)$ of a d–tensor:

$$N^{\cdot\underline{\kappa}}_{\underline{\tau}}(\underline{u}) = I_{\overline{S}_p} N^{\cdot\cdot\overline{\kappa}}_{\overline{\tau}\cdot\cdot\overline{\alpha}_1\ldots\overline{\alpha}_p}(\overline{u}) h^{\overline{\tau}}_{\underline{\tau}}(\underline{u},\overline{u}) h^{\underline{\kappa}}_{\overline{\kappa}}(\overline{u},\underline{u}) dS^{\overline{\alpha}_1\ldots\overline{\alpha}_p}.$$

We point out that tensor–integral can be defined not only for dc–tensors but and for d–tensors on ξ. Really, suppressing indices φ and γ in (3.42) and (3.43), considering instead of a deformation dc–tensor a deformation tensor

$$B^{\cdot\cdot\gamma}_{\alpha\beta}(u,\underline{u}) = B^{\cdot\cdot\gamma}_{\alpha\beta}(u) = P^{\gamma}_{.\alpha\beta}(u) \tag{3.46}$$

(we consider deformations induced by a nc–transform) and integration $I_{S_p} \ldots dS^{\alpha_1\ldots\alpha_p}$ in la–space ξ we obtain from (3.34) a tensor–integral on $\mathcal{C}(\xi)$ of a d–tensor:

$$N^{\cdot\underline{\kappa}}_{\underline{\tau}}(\underline{u}) = I_{S_p} N^{\cdot\kappa}_{\tau.\alpha_1\ldots\alpha_p}(u) h^{\tau}_{\underline{\tau}}(\underline{u},u) h^{\underline{\kappa}}_{\kappa}(u,\underline{u}) dS^{\alpha_1\ldots\alpha_p}.$$

Taking into account (3.38) we can calculate that curvature

$$\underline{R}^{\cdot\lambda}_{\gamma.\alpha\beta} = D_{[\beta} B^{\cdot\cdot\lambda}_{\alpha]\gamma} + B^{\cdot\cdot\tau}_{[\alpha|\gamma|} B^{\cdot\cdot\lambda}_{\beta]\tau} + T^{\tau\cdot\cdot}_{.\alpha\beta} B^{\cdot\cdot\lambda}_{\tau\gamma}$$

of connection $\underline{\Gamma}^{\gamma}_{.\alpha\beta}(u) = \Gamma^{\gamma}_{.\alpha\beta}(u) + B^{\cdot\cdot\gamma}_{\alpha\beta.}(u)$, with $B^{\cdot\cdot\gamma}_{\alpha\beta}(u)$ taken from (3.46), vanishes, $\underline{R}^{\cdot\lambda}_{\gamma.\alpha\beta} = 0$. So, we can conclude that la–space ξ admits a tensor integral structure on $\mathcal{C}(\xi)$ for d–tensors associated to deformation tensor $B^{\cdot\cdot\gamma}_{\alpha\beta}(u)$ if the nc–image $\underline{\xi}$ is locally parallelizable. That way we generalize the one space tensor integral constructions in [Gottlieb and Vacaru 1996], were the possibility to introduce tensor integral structure on a curved space was restricted by the condition that this space is locally parallelizable. For $q = n + m$ relations (3.43), written for d–tensor $\mathcal{N}^{\cdot\underline{\beta\gamma}}_{\underline{\alpha}}$ (we change indices $\overline{\alpha}, \overline{\beta}, \ldots$ into $\underline{\alpha}, \underline{\beta}, \ldots$) extend the Gauss formula on $\mathcal{C}(\xi)$:

$$I_{S_{q-1}} \mathcal{N}^{\cdot\underline{\beta\gamma}}_{\underline{\alpha}} d\mathcal{S}_{\underline{\gamma}} = I_{\underline{S}_q} \underline{{}^{q-1}} D_{\underline{\tau}} \mathcal{N}^{\cdot\underline{\beta\tau}}_{\underline{\alpha}} d\underline{V}, \tag{3.47}$$

where $d\underline{V} = \sqrt{|\underline{G}_{\alpha\beta}|} d\underline{u}^1 \ldots d\underline{u}^q$ and

$$\underline{{}^{q-1}} D_{\underline{\tau}} \mathcal{N}^{\cdot\underline{\beta\tau}}_{\underline{\alpha}} = D_{\underline{\tau}} \mathcal{N}^{\cdot\underline{\beta\tau}}_{\underline{\alpha}} - T^{\underline{\epsilon}}_{\cdot\underline{\tau\epsilon}} \mathcal{N}^{\underline{\beta\tau}}_{\underline{\alpha}} - B^{\cdot\cdot\underline{\epsilon}}_{\underline{\tau\alpha}} \mathcal{N}^{\cdot\underline{\beta\tau}}_{\underline{\epsilon}} + B^{\cdot\cdot\underline{\beta}}_{\underline{\tau\epsilon}} \mathcal{N}^{\cdot\underline{\epsilon\tau}}_{\underline{\alpha}}. \tag{3.48}$$

Let consider physical values $N^{\cdot\underline{\beta}}_{\underline{\alpha}}$ on $\underline{\xi}$ defined on its density $\mathcal{N}^{\cdot\underline{\beta\gamma}}_{\underline{\alpha}}$, i. e.

$$N^{\cdot\underline{\beta}}_{\underline{\alpha}} = I_{\underline{S}_{q-1}} \mathcal{N}^{\cdot\underline{\beta\gamma}}_{\underline{\alpha}} d\mathcal{S}_{\underline{\gamma}} \tag{3.49}$$

with this conservation law (due to (3.47)):

$$\underline{{}^{q-1}} D_{\underline{\gamma}} \mathcal{N}^{\cdot\underline{\beta\gamma}}_{\underline{\alpha}} = 0. \tag{3.50}$$

We note that these conservation laws differ from covariant conservation laws for well known physical values such as density of electric current or of energy– momentum tensor. For example, taking density $E^{\cdot\gamma}_{\beta}$, with corresponding to (3.48) and (3.50) conservation law,

$$\underline{{}^{q-1}} D_{\underline{\gamma}} E^{\underline{\gamma}}_{\underline{\beta}} = D_{\underline{\gamma}} E^{\underline{\gamma}}_{\underline{\beta}} - T^{\underline{\tau}}_{\cdot\underline{\epsilon\tau}} E^{\cdot\underline{\epsilon}}_{\underline{\beta}} - B^{\cdot\cdot\underline{\epsilon}}_{\underline{\tau\beta}} E^{\underline{\tau}}_{\underline{\epsilon}} = 0, \tag{3.51}$$

we can define values (see (3.47) and (3.49))

$$\mathcal{P}_{\alpha} = I_{\underline{S}_{q-1}} E^{\cdot\underline{\gamma}}_{\underline{\alpha}} d\mathcal{S}_{\underline{\gamma}}.$$

The defined conservation laws (3.51) for $E^{\cdot\underline{\epsilon}}_{\underline{\beta}}$ have nothing to do with those for energy–momentum tensor $E^{\cdot\gamma}_{\alpha}$ from Einstein equations for the almost Hermitian gravity [Miron and Anastasieie 1994] or with $\tilde{E}_{\alpha\beta}$ from (3.45) with vanishing divergence $D_{\gamma}\tilde{E}^{\cdot\gamma}_{\alpha} = 0$. So $\tilde{E}^{\cdot\gamma}_{\alpha} \neq E^{\cdot\gamma}_{\alpha}$. A similar conclusion was made in [Gottlieb and Vacaru 1996] for unispacial locally isotropic tensor integral. In the case of multispatial tensor integration we have another possibility, namely, to identify $E^{\cdot\underline{\gamma}}_{\underline{\beta}}$ from (3.51) with the na-image of $E^{\cdot\gamma}_{\beta}$ on la–space ξ. We shall consider this construction in the next section.

III.5 NA–Conservation Laws in LA–Gravity

It is well known that the standard pseudo–tensor description of the energy–momentum values for the Einstein gravitational fields is full of ambiguities. Some light can be shed by introducing additional geometrical structures on curved space–time (bimetrics, biconnections, by taking into account background spaces, or formulating variants of general relativity theory on flat space). We emphasize here that rigorous mathematical investigations based on two (fundamental and background) locally anisotropic, or isotropic, spaces should use well–defined, motivated from physical point of view, mappings of these spaces. Our na–model largely contains both attractive features of the mentioned approaches; na–maps establish a local 1–1 correspondence between the fundamental la–space and auxiliary la–spaces on which biconnection (or even multiconnection) structures are induced. But these structures are not a priory postulated as in a lot of gravitational theories, we tend to specify them to be locally reductible to the locally isotropic Einstein theory.

Let us consider a fixed background la–space $\underline{\xi}$ with given metric $\underline{G}_{\alpha\beta} = (\underline{g}_{ij}, \underline{h}_{ab})$ and d–connection $\tilde{\underline{\Gamma}}^{\alpha}_{\cdot\beta\gamma}$, for simplicity in this subsection we consider compatible metric and connections being torsionless and with vanishing curvatures. Supposing that there is an nc–transform from the fundamental la–space ξ to the auxiliary $\underline{\xi}$. we are interested in the equivalents of the Einstein equations (3.45) on $\underline{\xi}$.

We consider that a part of gravitational degrees of freedom is "pumped out" into the dynamics of deformation d–tensors for d–connection, $P^{\alpha}{}_{\beta\gamma}$, and metric, $B^{\alpha\beta} = (b^{ij}, b^{ab})$. The remained part of degrees of freedom is coded into the metric $\underline{G}_{\alpha\beta}$ and d–connection $\tilde{\underline{\Gamma}}^{\alpha}_{\cdot\beta\gamma}$.

We apply the first order formalism and consider $B^{\alpha\beta}$ and $P^{\alpha}{}_{\beta\gamma}$ as independent variables on $\underline{\xi}$. Using notations

$$P_{\alpha} = P^{\beta}{}_{\beta\alpha}, \quad \Gamma_{\alpha} = \Gamma^{\beta}{}_{\beta\alpha}, \ \hat{B}^{\alpha\beta} = \sqrt{|G|}B^{\alpha\beta}, \hat{G}^{\alpha\beta} = \sqrt{|G|}G^{\alpha\beta}, \hat{\underline{G}}^{\alpha\beta} = \sqrt{|\underline{G}|}\underline{G}^{\alpha\beta}$$

and making identifications

$$\hat{B}^{\alpha\beta} + \hat{\underline{G}}^{\alpha\beta} = \hat{G}^{\alpha\beta}, \quad \underline{\Gamma}^{\alpha}_{\cdot\beta\gamma} - P^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma},$$

we take the action of la–gravitational field on $\underline{\xi}$ in this form:

$$\underline{\mathcal{S}}^{(g)} = -(2c\kappa_1)^{-1} \int \delta^q u \underline{\mathcal{L}}^{(g)}, \tag{3.52}$$

where

$$\underline{\mathcal{L}}^{(g)} = \hat{B}^{\alpha\beta}(D_{\beta}P_{\alpha} - D_{\tau}P^{\tau}{}_{\alpha\beta}) + (\hat{\underline{G}}^{\alpha\beta} + \hat{B}^{\alpha\beta})(P_{\tau}P^{\tau}{}_{\alpha\beta} - P^{\alpha}{}_{\alpha\kappa}P^{\kappa}{}_{\beta\tau})$$

and the interaction constant is taken $\kappa_1 = \frac{4\pi}{c^4}k$, ($c$ is the light constant and k is Newton constant) in order to obtain concordance with the Einstein theory in the locally isotropic limit.

We construct on $\underline{\xi}$ a la–gravitational theory with matter fields (denoted as φ_A with A being a general index) interactions by postulating this Lagrangian density for matter fields

$$\underline{\mathcal{L}}^{(m)} = \underline{\mathcal{L}}^{(m)}[\underline{\hat{G}}^{\alpha\beta} + \hat{B}^{\alpha\beta}; \frac{\delta}{\delta u^{\gamma}}(\underline{\hat{G}}^{\alpha\beta} + \hat{B}^{\alpha\beta}); \varphi_A; \frac{\delta \varphi_A}{\delta u^{\tau}}]. \tag{3.53}$$

Starting from (3.52) and (3.53) the total action of la–gravity on $\underline{\xi}$ is written as

$$\underline{S} = (2c\kappa_1)^{-1} \int \delta^q u \underline{\mathcal{L}}^{(g)} + c^{-1} \int \delta^{(m)} \underline{\mathcal{L}}^{(m)}. \tag{3.54}$$

Applying variational procedure on $\underline{\xi}$, locally adapted to N–connection by using derivations (1.4) instead of partial derivations, we derive from (3.54) the la–gravitational field equations

$$\Theta_{\alpha\beta} = \kappa_1(\underline{\mathbf{t}}_{\alpha\beta} + \underline{\mathbf{T}}_{\alpha\beta}) \tag{3.55}$$

and matter field equations

$$\frac{\triangle \underline{\mathcal{L}}^{(m)}}{\triangle \varphi_A} = 0, \tag{3.56}$$

where $\frac{\triangle}{\triangle \varphi_A}$ denotes the variational derivation.

In (3.55) we have introduced these values: the energy–momentum d–tensor for la–gravitational field

$$\kappa_1 \underline{\mathbf{t}}_{\alpha\beta} = (\sqrt{|G|})^{-1} \frac{\triangle \underline{\mathcal{L}}^{(g)}}{\triangle G^{\alpha\beta}} = K_{\alpha\beta} + P^{\gamma}{}_{\alpha\beta} P_{\gamma} - P^{\gamma}{}_{\alpha\tau} P^{\tau}{}_{\beta\gamma} + \frac{1}{2} \underline{G}_{\alpha\beta} \underline{G}^{\gamma\tau} (P^{\phi}{}_{\gamma\tau} P_{\phi} - P^{\phi}{}_{\gamma\epsilon} P^{\epsilon}{}_{\phi\tau}), \tag{3.57}$$

(where

$$K_{\alpha\beta} = \underline{D}_{\gamma} K^{\gamma}_{\alpha\beta},$$

$$2K^{\gamma}_{\alpha\beta} = -B^{\tau\gamma} P^{\epsilon}{}_{\tau(\alpha} \underline{G}_{\beta)\epsilon} - B^{\tau\epsilon} P^{\gamma}{}_{\epsilon(\alpha} \underline{G}_{\beta)\tau} + \underline{G}^{\gamma\epsilon} h_{\epsilon(\alpha} P_{\beta)} + \underline{G}^{\gamma\tau} \underline{G}^{\epsilon\phi} P^{\varphi}{}_{\phi\tau} \underline{G}_{\varphi(\alpha} B_{\beta)\epsilon} + \underline{G}_{\alpha\beta} B^{\tau\epsilon} P^{\gamma}{}_{\tau\epsilon} - B_{\alpha\beta} P^{\gamma} \quad),$$

$$2\Theta = \underline{D}^{\tau} \underline{D}_{tau} B_{\alpha\beta} + \underline{G}_{\alpha\beta} \underline{D}^{\tau} \underline{D}^{\epsilon} B_{\tau\epsilon} - \underline{G}^{\tau\epsilon} \underline{D}_{\epsilon} \underline{D}_{(\alpha} B_{\beta)\tau}$$

and the energy–momentum d–tensor of matter

$$\underline{\mathbf{T}}_{\alpha\beta} = 2 \frac{\triangle \mathcal{L}^{(m)}}{\triangle \underline{\hat{G}}^{\alpha\beta}} - \underline{G}_{\alpha\beta} \underline{G}^{\gamma\delta} \frac{\triangle \mathcal{L}^{(m)}}{\triangle \underline{\hat{G}}^{\gamma\delta}}. \tag{3.58}$$

As a consequence of (3.56)–(3.58) we obtain the d–covariant on $\underline{\xi}$ conservation laws

$$\underline{D}_{\alpha}(\underline{\mathbf{t}}^{\alpha\beta} + \underline{\mathbf{T}}^{\alpha\beta}) = 0. \tag{3.59}$$

We have postulated the Lagrangian density of matter fields (3.53) in a form as to treat $\underline{\mathbf{t}}^{\alpha\beta} + \underline{\mathbf{T}}^{\alpha\beta}$ as the source in (3.55).

Now we formulate the main results of this section:

Proposition 3.1. *The dynamics of the Einstein la–gravitational fields, modeled as solutions of equations (3.45) and matter fields on la–space* ξ*, can be equivalently locally modeled on a background la–space* $\underline{\xi}$ *provided with a trivial d-connection and metric structures having zero d–tensors of torsion and curvature by field equations (3.55) and (3.56) on condition that deformation tensor* $P^{\alpha}{}_{\beta\gamma}$ *is a solution of the Cauchy problem posed for basic equations for a chain of na–maps from* ξ *to* $\underline{\xi}$*.*

Proposition 3.2. *Local, d–tensor, conservation laws for Einstein la–gravitational fields can be written in form (3.59) for la–gravitational (3.57) and matter (3.58) energy–momentum d–tensors. These laws are d–covariant on the background space* $\underline{\xi}$ *and must be completed with invariant conditions of type (3.12)–(3.15) for every deformation parameters of a chain of na–maps from* ξ *to* $\underline{\xi}$*.*

The above presented considerations consist proofs of both propositions.

We emphasize that nonlocalization of both locally anisotropic and isotropic gravitational energy–momentum values on the fundamental (locally anisotropic or isotropic) space ξ is a consequence of the absence of global group automorphisms for generic curved spaces. Considering gravitational theories from view of multispaces and their mutual maps (directed by the basic geometric structures on ξ such as N–connection, d–connection, d–torsion and d–curvature components, see coefficients for basic na–equations (3.9)–(3.11)), we can formulate local d–tensor conservation laws on auxiliary globally automorphic spaces being related with space ξ by means of chains of na–maps. Finally, we remark that as a matter of principle we can use d–connection deformations in order to modelate the la–gravitational interactions with nonvanishing torsion and nonmetricity. In this case we must introduce a corresponding source in (3.59) and define generalized conservation laws as in (3.44) (see similar details for locally isotropic generalizations of the Einstein gravity in [Vacaru 1994]).

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